Möbius inversion

Recall: If P is any poset, then $\mu = \mu_P$ is uniquely defined by

$$\cdot\ \mu(x,y) = 0 \quad \text{if} \quad x \not\leq y$$

$$\cdot\ \sum_{x \leq z \leq y} \mu(x,z) = \delta_{xy} \quad \text{if} \quad x \leq y$$

where $\delta_{xy} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$.

Lemma: If $x \leq y$ in P, then $\sum_{x \leq z \leq y} \mu(z,y) = \delta_{xy}$.

Proof: Define ν by

$$\cdot\ \nu(x,y) = 0 \quad \text{if} \quad x \not\leq y$$

$$\cdot\ \sum_{x \leq z \leq y} \nu(z,y) = \delta_{xy} \quad \text{if} \quad x \leq y.$$

Claim: $\nu = \mu$.

Indeed, if $x \le y$, then

$$\nu(x,y) = \sum_{x \le z \le y} \nu(z,y) \cdot \delta_{xz}$$

$$= \sum_{x \le z \le y} \nu(z,y) \cdot \sum_{x \le w \le z} \mu(x,w).$$

Swap the order of summation:

$$\nu(x,y) = \sum_{x \le w \le y} \mu(x,w) \sum_{w \le z \le y} \nu(z,y)$$

$$= \sum_{x \le w \le y} \mu(x,w) \cdot \delta_{wy}$$

$$= \mu(x,y).$$ ∎

<u>Thm</u> (Möbius inversion): Let P be a finite poset, G an abelian group, and $f, g : P \to G$ functions. Then

$$f(x) = \sum_{y \ge x} g(y) \iff g(x) = \sum_{y \ge x} \mu(x,y) f(y)$$

Dually,

$$f(x) = \sum_{y \le x} g(y) \iff g(x) = \sum_{y \le x} \mu(y,x) f(y)$$

<u>Proof</u>: If $f(x) = \sum_{y \ge x} g(y)$, then

$$\sum_{y \ge x} \mu(x,y) f(y) = \sum_{y \ge x} \mu(x,y) \sum_{z \ge y} g(z)$$

$$= \sum_{z \ge x} g(z) \sum_{x \le y \le z} \mu(x,y)$$

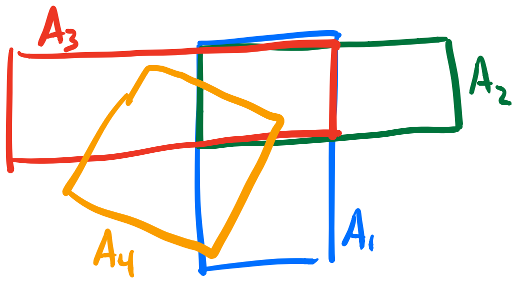
$$= \sum_{z \ge x} g(z) \, \delta_{xz}$$

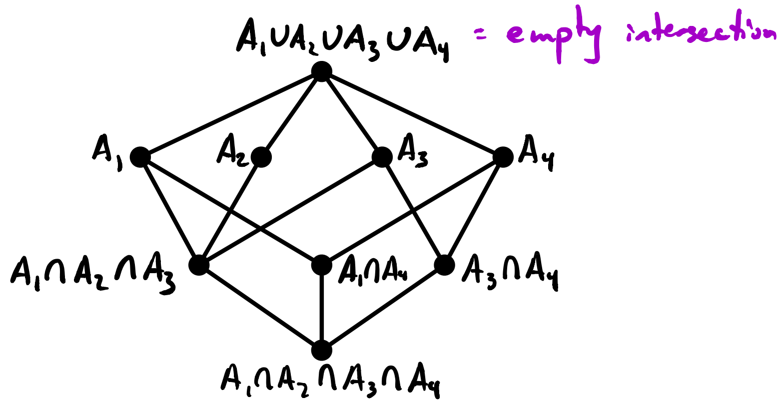
$$= g(x).$$

The other parts are similar. $\blacksquare$

Möbius inversion generalizes inclusion-exclusion

Let A_i be finitely many finite sets,
P = intersection poset.





Let

$$g : \mathcal{P} \longrightarrow \mathbb{Z}_{\geq 0}$$

$S \longmapsto$ # elements in S which are not in any $S' \in \mathcal{P}$ with $S' \subsetneq S$.

and

$$f : \mathcal{P} \longrightarrow \mathbb{Z}_{\geq 0}$$

$$S \longmapsto \sum_{T \subseteq S} g(T) = |S|$$

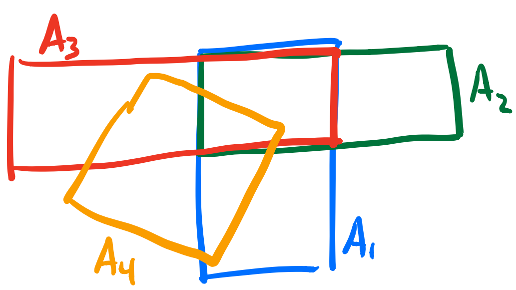
To compute $|\bigcup_i A_i| = f(\bigcup_i A_i)$, use Möbius inversion:

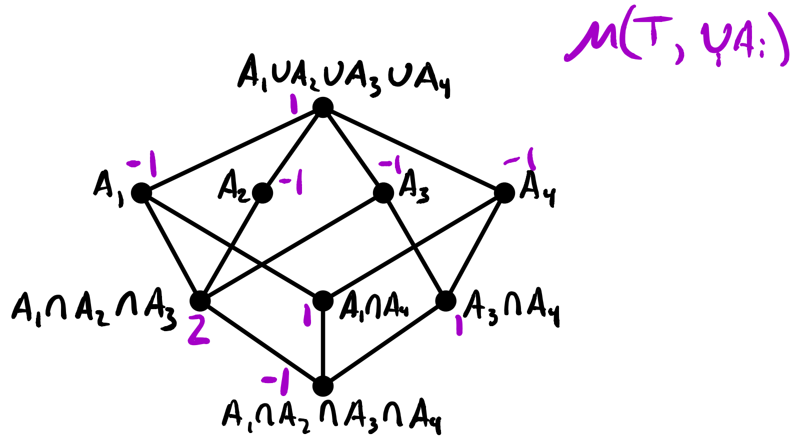
$$g(\bigcup_i A_i) = \sum_{T \subseteq \bigcup_i A_i} \mu(T, \bigcup_i A_i)\, f(T)$$

$$\Rightarrow \quad 0 = \sum_{T \in \mathcal{P}} \mu(T, \bigcup_i A_i)\, |T|$$

$$\Rightarrow \boxed{\left|\bigcup_i A_i\right| = -\sum_{\substack{T\in \mathcal{P}\\ T\neq \bigcup_i A_i}} \mu\left(T, \bigcup_i A_i\right)|T|.}$$

Ex: In the picture above





$$\rightarrow |A_1 \cup A_2 \cup A_3 \cup A_4|$$

$$= |A_1| + |A_2| + |A_3| + |A_4|$$

$$- 2|A_1 \cap A_2 \cap A_3| - |A_1 \cap A_4| - |A_3 \cap A_4|$$

$$+ |A_1 \cap A_2 \cap A_3 \cap A_4|.$$