Last time: Möbius inversion for counting $|\bigcup_i A_i|$.

A similar idea: Point-counting over $\mathbb{F}_q$.

Let $M = M(\mathcal{A})$, where $\mathcal{A}$ is a configuration $\{v_e \mid e \in E\}$ in an $\mathbb{F}_q$-vector space V.

WLOG, $\mathcal{A}$ spans V, so

$$\dim V = \mathrm{rk}(M) =: r \qquad \Rightarrow |V| = q^r$$

We get an associated hyperplane arrangement in V^*, where

$$v_e \longrightarrow H_e = \{ f \in V^* \mid f(v_e) = 0 \}.$$

cf. Lecture 12 Exercise 1

Notation: For $X \subseteq E$, set $H_X = \bigcap_{e \in X} H_e$, so that

$$\begin{aligned} \mathrm{rk}(X) &= \mathrm{codim}(H_X) \\ &= r - \dim(H_X). \end{aligned}$$

Then

$$\bullet\ H_X = H_{cl(X)}.$$

$$\bullet\ F \text{ is a flat} \iff H_{F\cup e} = H_F \cap H_e \subsetneq H_F \text{ for all } e \notin F.$$

That is

flats $\leftrightarrow$ subspaces of V^* obtained by intersecting the H_e.

For each flat F, let

$$g(F) = \Big| H_F \setminus \Big(\bigcup_{\substack{G \text{ flat} \\ G \supsetneq F}} H_G \Big) \Big|$$

= # points in H_F which aren't contained in any $H_G \subsetneq H_F$.

Then

$$f(F) = \sum_{G \supseteq F} g(G) = |H_F| = q^{crk(F)}.$$

a subspace of dimension $r - rk(F)$

By Möbius inversion,

$$g(F) = \sum_{G \supseteq F} \mu(F,G)\, f(G).$$

In particular (assuming M is loopless),

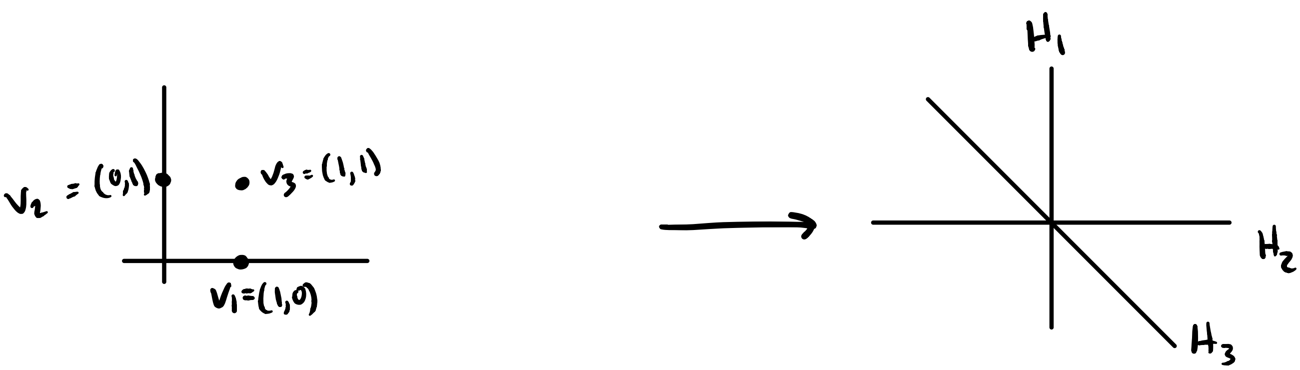
$$g(\emptyset) = \sum_{G \in \mathcal{L}(M)} \mu(\emptyset, G)\, q^{\mathrm{crk}(G)}$$

i.e.

$$\left| V^* \setminus \bigcup_{e \in E} H_e \right| = \chi_M(q)$$

The characteristic polynomial counts the complement of the hyperplane arr. (over $\mathbb{F}_q$).

Ex: $M = U_{2,3}$

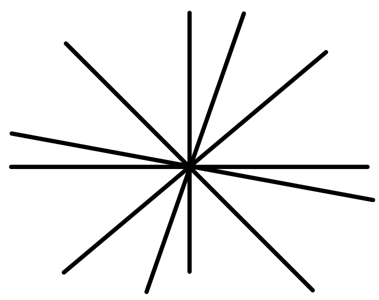


$\mathcal{A}$ in $\mathbb{F}_q^2$

$(\mathbb{F}_q^2)^*$

$$\Rightarrow \chi_{U_{2,3}}(q) = q^2 - 3q + 2$$

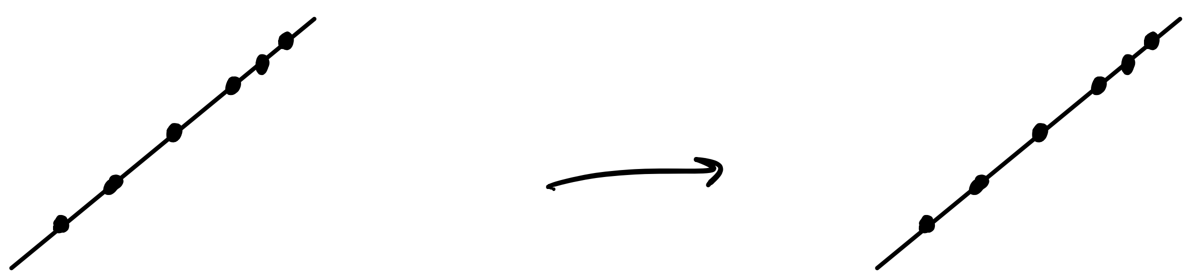
Ex: $U_{2,n}$ is represented by n distinct lines in $(\mathbb{F}_q^2)^*$



$$\Rightarrow \quad \chi_{U_{2,n}}(q) = q^2 - nq + (n-1)$$

Exact same reasoning: $\overline{\chi}_M(q) = \dfrac{\chi_M(q)}{q-1}$ counts the complement of the projectivized hyp. arr.

Ex: $U_{2,n}$

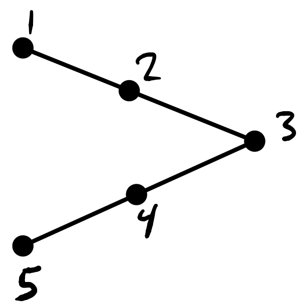


$\mathbb{P}\mathcal{A}$ in $\mathbb{P}^1_{\mathbb{F}_q}$ $\qquad\qquad$ $(\mathbb{P}^1_{\mathbb{F}_q})^\vee$

$$\overline{\chi}_{U_{2,n}}(q) = (q+1) - n$$

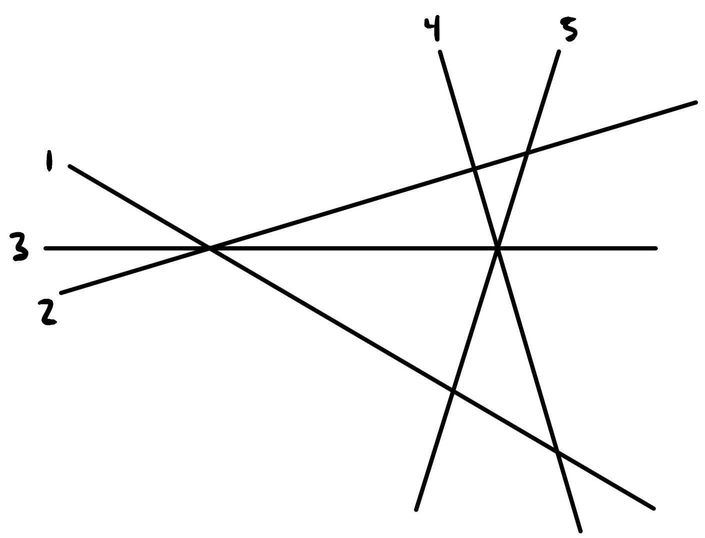
$$= q - (n-1)$$

Ex: Let M be the matroid with geometric rep.



$\subseteq \mathbb{P}^2$

The corresponding projective hyperplane arr. is



$\subseteq (\mathbb{P}^2)^\vee$

So

$$\overline{\chi}_M(q) = \underbrace{q^2+q+1}_{=|\mathbb{P}^2| = \frac{q^3-1}{q-1}} - 5\underbrace{(q+1)}_{=|\mathbb{P}^1| = \frac{q^2-1}{q-1}} + 4\cdot 1 + 2\cdot 2$$

$$= q^2 - 4q + 4$$

$$\Rightarrow \chi_M(q) = (q-1)(q^2-4q+4) = q^3-5q^2+8q-4.$$