Recall: The ith Whitney number of the first kind for a loopless matroid M is

$$w_i(M) = w_i = \text{coefficient of } q^{rk(M)-i} \text{ in } \chi_M$$

$$= \sum_{\substack{F \in \mathcal{L}(M) \\ rk(F) = i}} \mu(\emptyset, F).$$

$\hookrightarrow F \in \mathcal{L}(M)_i$

Today: Compute these numbers faster.

Thm (Weisner): Let M be a loopless matroid on E. If F is a non-empty flat of M, then

$$\sum_{\substack{G \in \mathcal{L}(M) \\ F \vee G = E}} \mu(\emptyset, G) = 0.$$

Recall: $F \vee G = cl(F \cup G)$

Proof:

$$\sum_{\substack{G \in \mathcal{L}(M) \\ F \vee G = E}} \mu(\emptyset, G) = \sum_{G \in \mathcal{L}(M)} \mu(\emptyset, G) \cdot \delta_{F \vee G, E}$$

$$= \sum_{G \in \mathcal{L}(M)} \mu(\emptyset, G) \cdot \sum_{\substack{H \\ F \vee G \subseteq H \subseteq E}} \mu(H, E)$$

$F \vee G \subseteq H \iff F \subseteq H$ and $G \subseteq H$

$$= \sum_{\substack{H \in \mathcal{L}(M) \\ H \geq F}} \mu(H, E) \sum_{\substack{G \\ \emptyset \subseteq G \subseteq H}} \mu(\emptyset, G)$$

$$= \sum_{\substack{H \\ H \geq F}} \mu(H, E)\, \delta_{\emptyset, H}$$

$$= 0$$

since $H \geq F \gneq \emptyset$ in each term. $\blacksquare$

Cor: Let M be a loopless matroid on E, and fix a rank 1 flat A. Then

$$\mu(\emptyset, E) = - \sum_{\substack{H \in \mathcal{H}(M) \\ \text{s.t. } A \nsubseteq H}} \mu(\emptyset, H).$$

More generally, if F is a nonempty flat, and A is a fixed rank 1 flat with $A \subseteq F$, then

$$\mu(\emptyset, F) = - \sum_{\substack{H \lessdot F \\ \text{s.t. } A \nsubseteq H}} \mu(\emptyset, H).$$

Recall: $H \lessdot F$ means F covers H.

Proof: The second statement follows by applying the first to MIF.

Let G be any flat such that $A \vee G = E$.
If $A \subseteq G$, then $A \vee G = G$, so $G = E$.

Otherwise, $A \nsubseteq G$, so $A \cap G$ is properly contained in A. So $A \cap G = \emptyset$.

By submodularity,

$$\mathrm{rk}(\underbrace{A \vee G}_{=E}) + \mathrm{rk}(\underbrace{A \wedge G}_{=\emptyset}) \leq \mathrm{rk}(A) + \mathrm{rk}(G)$$

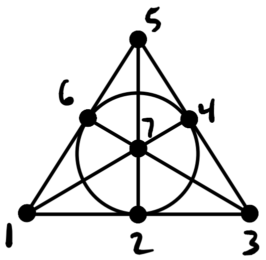
$$\Rightarrow \quad \mathrm{rk}(M) + 0 \leq 1 + \mathrm{rk}(G)$$

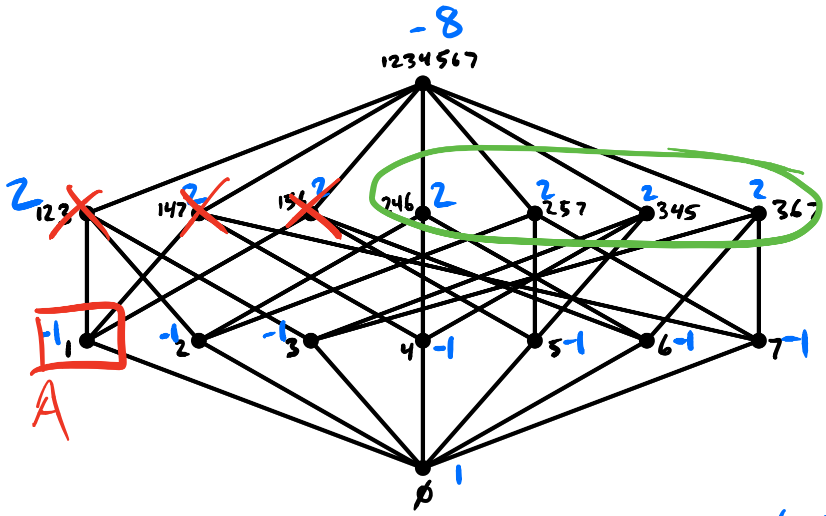
$$\Rightarrow \quad \mathrm{rk}(G) \geq \mathrm{rk}(M) - 1.$$

Since $G \neq E$, we have equality and G is a hyperplane.
By Weisner,

$$0 = \sum_{\substack{G \\ A \vee G = E}} \mu(\emptyset, G) = \mu(\emptyset, E) + \sum_{\substack{H \in \mathcal{H}(M) \\ A \nsubseteq H}} \mu(\emptyset, H).$$

Ex: $M = F_7$





$\mu(\emptyset, F)$

$$\mu(\emptyset,F) = -\sum_{\substack{H \subsetneq F \\ \text{s.t. } A \not\subseteq H}} \mu(\emptyset,H)$$

$A \leq F$

Note on hypotheses:

- Weisner's theorem holds for any finite lattice
- The corollary holds for any finite submodular lattice.