## Last time:

**Cor:** Let $M$ be a loopless matroid on $E$, and fix a rank 1 flat $A$. Then

$$\mu(\emptyset, E) = - \sum_{\substack{H \in \mathcal{H}(M) \\ \text{s.t. } A \notin H}} \mu(\emptyset, H).$$

More generally, if $F$ is a nonempty flat of $M$, and $A$ is a fixed rank 1 flat with $A \subseteq F$, then

$$\mu(\emptyset, F) = - \sum_{\substack{H \subseteq F \\ \text{s.t. } A \notin H}} \mu(\emptyset, H).$$

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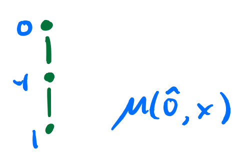
We already know that

$$(-1)^i w_i = \sum_{\substack{F \\ rk(F) = i}} (-1)^i \mu(\emptyset, F) > 0.$$

Actually, every term in this sum is positive.

**Cor** (Rota): For flats $F \subsetneq G$ of a loopless matroid $M$,

$$(-1)^{rk(G) - rk(F)} \mu(F, G) > 0.$$



**Proof:** It suffices to prove $(-1)^{rk(M)} \mu(\emptyset, E) > 0$, because $\mathcal{L}(M|G/F)$ is the interval $[F,G]$ in $\mathcal{L}(M)$.

$[F,G] = \{H \mid F \subseteq H \subseteq G\}$

If $rk(M) = 1$, then $E$ covers $\emptyset$, and


$$\mu(\emptyset, E) = -1 \quad ✓$$

Otherwise, fix an atom $A$. By the corollary,

$$(-1)^{rk(M)} \mu(\emptyset, E) = (-1)^{rk(M)} \Big( - \sum_{\substack{H \in \mathcal{H}(M) \\ A \notin H}} \mu(\emptyset, H) \Big)$$

$$= \sum_{\substack{H \in \mathcal{H}(M) \\ A \notin H}} (-1)^{rk(H)} \mu(\emptyset, H)$$

$$> 0$$

by induction. ∎

# Coefficients of $\overline{\chi}_M$

Let $M$ be a loopless matroid of rank $r > 0$.

Recall

$$\chi_M(q) = \sum_{i=0}^{r} w_i \, q^{r-i} \qquad w_i = \sum_{F \in \mathcal{L}(M)_i} \mu(\emptyset, F)$$

$$= \sum_{i=0}^{r} (-1)^i |w_i| \, q^{r-i}.$$

Write

$$\overline{\chi}_M(q) = \frac{\chi_M(q)}{q-1}$$

$$= \sum_{i=0}^{r-1} (-1)^i \mu^i \, q^{r-1-i}$$

Then

- $\mu^i > 0$ for all $0 \le i \le r-1$
- $\mu^i = w_i + w_{i-1} + \cdots + w_0$

  $= |w_i| - |w_{i-1}| + \cdots + (-1)^i |w_0|$
- $|w_i| = \mu^i + \mu^{i-1}$ (set $\mu^{-1} = 0 = \mu^r$)

By Lecture 22 Exercise 1, log-concavity of the $\mu^i$ implies log-concavity of the $|w_i|$.

Ex: $\chi_{U_{3,4}}(q) = q^3 - 4q^2 + 6q - 3$

$(|w_i|) = (1, 4, 6, 3)$

$\overline{\chi}_{U_{3,4}}(q) = q^2 - 3q + 3$

$(\mu^i) = (1, 3, 3)$

**Lemma:** Fix a rank 1 flat A. Then

$$\mu^i = (-1)^i \sum_{\substack{F \in \mathcal{L}(M)_i \\ A \notin F}} \mu(\emptyset, F) \quad \leftarrow \text{consistent with } \mu^r = 0$$

$$= (-1)^{i+1} \sum_{\substack{G \in \mathcal{L}(M)_{i+1} \\ A \subseteq G}} \mu(\emptyset, G) \quad \leftarrow \text{consistent with } \mu^{-1} = 0$$

**Proof:** We first show the sums are equal.
By Cor to Weisner:

$$\sum_{\substack{G \in \mathcal{L}(M)_{i+1} \\ A \subseteq G}} \mu(\emptyset, G) = \sum_{\substack{G \in \mathcal{L}(M)_{i+1} \\ A \subseteq G}} \left( - \sum_{\substack{F \subset G \\ A \notin F}} \mu(\emptyset, F) \right)$$

$$= -\sum_{\substack{F \in \mathcal{L}(M): \\ A \notin F}} \sum_{\substack{G \supseteq F \\ A \subseteq G}} \mu(\emptyset, F)$$

only one such $G$, namely $F \vee A = \mathrm{cl}(F \cup A)$

$$= -\sum_{\substack{F \in \mathcal{L}(M): \\ A \notin F}} \mu(\emptyset, F).$$

We now show this sum is $\mu^i$ by induction.

Clear when $i = 0$ ($\mu^0 = 1$).

If it's true for $i-1$, then

$$|W_i| = \mu^i + \mu^{i-1}$$

$$\Rightarrow \mu^i = |W_i| - \mu^{i-1}$$

$$= (-1)^i \sum_{F \in \mathcal{L}(M):} \mu(\emptyset, F) - (-1)^i \sum_{\substack{G \in \mathcal{L}(M): \\ A \subseteq G}} \mu(\emptyset, G)$$

$$= (-1)^i \sum_{\substack{F \in \mathcal{L}(M): \\ A \notin F}} \mu(\emptyset, F).$$

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