<u>Last time</u>: $\overline{\chi}_M(q) = \dfrac{\chi_M(q)}{q-1} = \sum_{i=0}^{r-1} (-1)^i \mu^i q^{r-1-i}$.

<u>Lemma</u>: Fix a rank 1 flat A. Then

$$\mu^i = (-1)^i \sum_{\substack{F \\ rk(F)=i \\ A \not\subseteq F}} \mu(\emptyset, F)$$

$$= (-1)^{i+1} \sum_{\substack{G \\ rk(G)=i+1 \\ A \subseteq G}} \mu(\emptyset, G)$$

A combinatorial formula for μ^i

WLOG, the ground set of M is $E = [n] = \{1, \ldots, n\}$.

<u>Def</u>: Let M be a loopless matroid on $[n]$. A <u>k-step flag of flats</u> in M is a chain

$$\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$$

with each F_i a flat of M.

- The flag is <u>initial</u> if $F_{i-1} \subset F_i$ for all i

$\iff \operatorname{rk}(F_i) = i$ for all i.

- The flag is <u>descending</u> if

$$\min(F_1) > \min(F_2) > \cdots > \min(F_k) > 1.$$

<u>Thm</u>: μ^k is the number of initial descending flags of flats in M.

<u>Proof</u>: Take $A = \operatorname{cl}(1)$ in the lemma:

$$\mu^k = (-1)^k \sum_{\substack{F_k \in \mathcal{L}(M)_k \\ 1 \notin F_k}} \mu(\emptyset, F_k)$$

Now, repeatedly use the corollary to Weisner's Theorem:

$$\mu^k = (-1)^k \sum_{\substack{F_k \in \mathcal{L}(M)_k \\ 1 \notin F_k}} \left(- \sum_{\substack{F_{k-1} \subset F_k \\ \min(F_k) \notin F_{k-1}}} \mu(\emptyset, F_{k-1}) \right)$$

$$= (-1)^{k-1} \sum_{F_k \in \mathcal{L}(M)_k} \sum_{F_{k-1} \subset F_k} \left(- \sum_{F_{k-2} \subset F_{k-1}} \mu(\emptyset, F_{k-2}) \right)$$

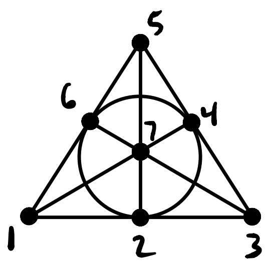
$1 \notin F_k \qquad \min(F_k) \notin F_{k-1} \qquad \min(F_{k-1}) \notin F_{k-2}$

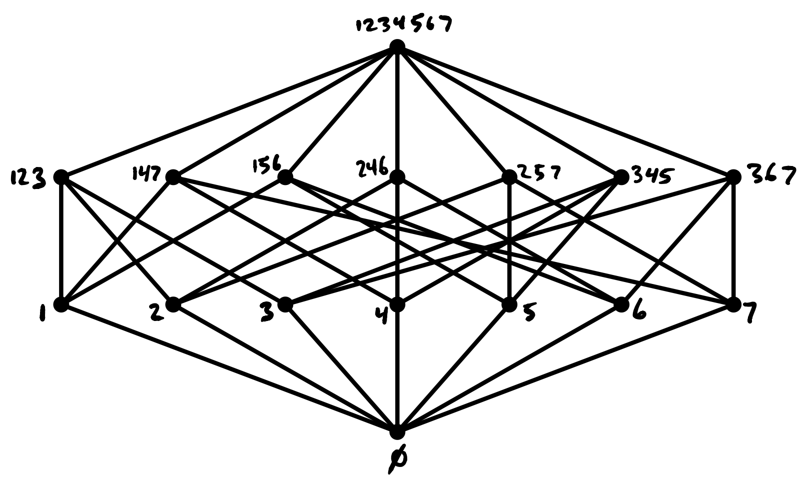
$\vdots$

$$= \cancel{(-1)} \sum_{\substack{F_k \supset F_{k-1} \supset F_{k-2} \supset \cdots \supset F_2 \supset F_1 \\ 1 < \min(F_k) < \min(F_{k-1}) < \cdots < \min(F_2) < \min(F_1)}} \cancel{\mu(\emptyset, F_1)} \; -1$$

= # of initial, descending flags. $\blacksquare$

Ex: $M = F_7$





Initial descending flags

0-step : $\emptyset$

1-step : $\emptyset < i$, $i = 2, \dots, 7$

2-step : $\emptyset < i < cl\{i, j\}$ $\quad i = 4, 5, 6, 7$, $j = 2, 3$

3-step : None

$\Rightarrow \mu^0 = 1$, $\mu^1 = 6$, $\mu^2 = 8$

$\Rightarrow \overline{\chi}_{F_7}(q) = q^2 - 6q + 8$ ✓

The topology of an arrangement complement

- M be a simple $\mathbb{C}$-representable matroid
- $\mathcal{A} = \{v_e \mid e \in E\}$ a configuration in a $\mathbb{C}$-vector space V realizing M (WLOG $\mathcal{A}$ spans V).
- $\{H_e \mid e \in E\}$ the associated hyperplane arrangement in V^*.

__Question__: which properties of this hyperplane arrangement depend only on M?

Often, we think in terms of the complement

$$U_{\mathcal{A}} := V^* \setminus \bigcup_{e \in E} H_e$$

If $M(\mathcal{A}_1) = M(\mathcal{A}_2)$, then how do $U_{\mathcal{A}_1}$ and $U_{\mathcal{A}_2}$ compare?