The topology of an arrangement complement

Let

- $\mathcal{A} = \{v_e \mid e \in E\}$ a configuration in a $\mathbb{C}$-vector space V (WLOG $\mathcal{A}$ spans V).

- $\{H_e \mid e \in E\}$ the associated hyperplane arrangement in V^*.

Let

$$U_{\mathcal{A}} := V^* \setminus \bigcup_{e \in E} H_e .$$

<u>Big Project</u>: Compare $U_{\mathcal{A}_1}$ and $U_{\mathcal{A}_2}$ when
$$M(\mathcal{A}_1) = M(\mathcal{A}_2) = M.$$

<u>Easy</u>: $\dim U_{\mathcal{A}_1} = \dim U_{\mathcal{A}_2} = \mathrm{rk}(M)$.

<u>Orlik-Solomon '80</u>: $H^\bullet(U_{\mathcal{A}_1}; \mathbb{Z}) \cong H^\bullet(U_{\mathcal{A}_2}; \mathbb{Z})$

<u>Rybnikov, '94</u>: There exist arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ of 13 planes each in $\mathbb{C}^3$ such that $M(\mathcal{A}_1) = M(\mathcal{A}_2)$ but
$$\pi_1(U_{\mathcal{A}_1}) \not\cong \pi_1(U_{\mathcal{A}_2}).$$

Let $M = M(\mathcal{A})$ be a $\mathbb{C}$-representable matroid on $[n]$. Let's try to understand $H^\bullet(U_\mathcal{A};\mathbb{Z}) =: H^\bullet(U_\mathcal{A})$.

<u>Generators</u>: For each i, we have

$$U_\mathcal{A} \hookrightarrow V^* \setminus H_i \simeq \mathbb{C}^\times \simeq S^1$$

Let $\beta_i \in H^\bullet(U_\mathcal{A})$ be the pullback of the generator of $H^\bullet(S^1)$.

<u>Orlik-Solomon algebra</u>

Let $\Lambda^\bullet[x_1, \ldots, x_n]$ be graded exterior algebra (over $\mathbb{Z}$), with each generator in degree 1.

For $S = \{i_1, \ldots, i_k\} \subseteq [n]$ with $i_1 < i_2 < \cdots < i_k$, let

$$x_S := x_{i_1} \wedge \cdots \wedge x_{i_k} \quad \in \Lambda^\bullet[x_1, \ldots, x_n]$$

and

$$\partial x_S := \sum_{j=1}^{k} (-1)^{j-1} x_{S \setminus j}$$

<u>Ex</u>: $x_{123} = x_1 \wedge x_2 \wedge x_3 \in \Lambda^3[x_1, x_2, x_3]$

$$\partial x_{123} = x_{23} - x_{13} + x_{12} = x_2 \wedge x_3 - x_1 \wedge x_3 + x_1 \wedge x_2 \in \Lambda^2[x_1, x_2, x_3]$$

Def: The Orlik-Solomon ideal of the matroid M is

$$I_{OS} = \left(\partial x_C \mid C \in \mathcal{C}(M) \right) \subseteq \Lambda^\bullet[x_1, \dots, x_n]$$

The Orlik-Solomon algebra of M is the quotient

$$OS^\bullet(M) = \Lambda^\bullet[x_1, \dots, x_n] \,/\, I_{OS}$$

Note:
- I_{OS} is homogeneous $\Rightarrow$ $OS^\bullet(M)$ inherits the grading from $\Lambda^\bullet[x_1, \dots, x_n]$.
- $\partial I_{OS} \subseteq I_{OS}$, so $OS^\bullet(M)$ inherits the derivation ∂.

  (use graded Leibniz rule)

  $\hookrightarrow \partial(y \cdot \partial x_C) = 0$

Thm (Brieskorn, Orlik-Solomon '80): Let M be a simple $\mathbb{C}$-representable matroid, and let $\mathcal{A}$ be a configuration with $M = M(\mathcal{A})$. There is an isomorphism

$$\begin{aligned} OS^\bullet(M) &\longrightarrow H^\bullet(U_{\mathcal{A}}). \\ x_i &\longmapsto \beta_i \end{aligned}$$

Sketch: To show this is well-defined, we need

- $\beta_i^2 = 0$ $\quad$ ($H^2(S^1) = 0$)

- $\beta_i \beta_j = -\beta_j \beta_i$ $\quad$ (graded commutativity)

Define, for $S \subseteq [n]$, β_S and $\partial \beta_S \in H^\bullet(U_{\mathcal{A}})$ analogously to how we defined x_S and ∂x_S.

- If C is a circuit, then $\partial \beta_C = 0$

(Exercise in (de Rham) cohomology)