No class Friday

Thm (Brieskorn, Orlik-Solomon): Let M be a simple $\mathbb{C}$-representable matroid, and $\mathcal{A}$ a configuration with $M = M(\mathcal{A})$. There is an isomorphism

$$OS^\bullet(M) \longrightarrow H^\bullet(U_{\mathcal{A}})$$
$$x_i \longmapsto \beta_i$$

Proof sketch: Well-defined ✓

We now proceed by induction on n, the size of the ground set.

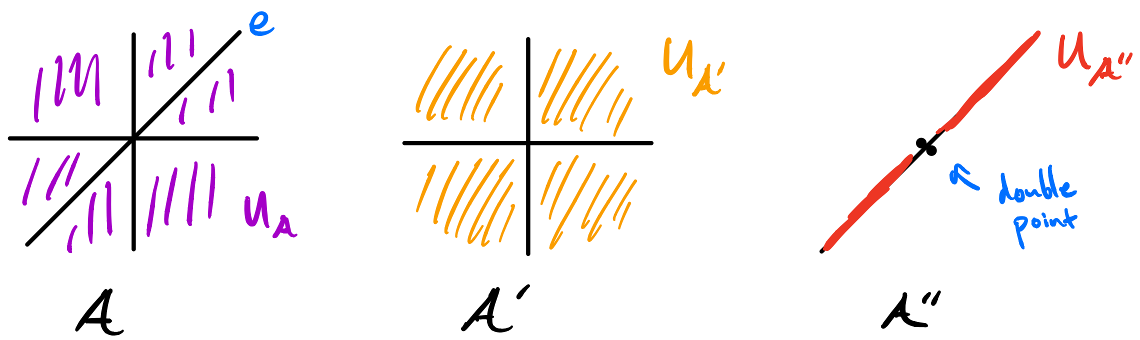
If $M = U_{0,0}$, then $OS^\bullet(M) = \mathbb{Z} = H^\bullet(pt)$.

Otherwise, we pick an element $e \in [n]$ and delete/contract.

Matroid	Vector configuration	Hyperplane arrangement
M	$\mathcal{A}$ in V	$\{H_i\}$ in V^*
$M\setminus e$	$\mathcal{A}\setminus\{v_e\}$ in V	$\{H_i \mid i\neq e\}$ in V^*
M/e	images of $\mathcal{A}\setminus\{v_e\}$ in $V/_{\mathrm{span}\{v_e\}}$	$\{H_i\cap H_e \mid i\neq e\}$ in H_e

For ease of notation, let $\mathcal{A}'$ and $\mathcal{A}''$ be the configurations corresponding to the deletion $M\setminus e$ and contraction M/e, resp.

Picture:



Then

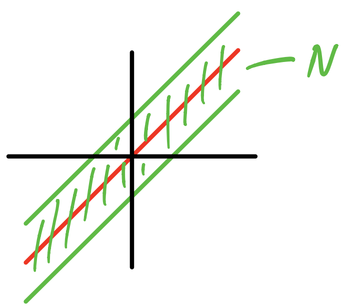
- $U_{\mathcal{A}},\ U_{\mathcal{A}''} \subset U_{\mathcal{A}'}$
- $U_{\mathcal{A}''} = U_{\mathcal{A}'} \cap H_e$
- $U_{\mathcal{A}} = U_{\mathcal{A}'} \setminus U_{\mathcal{A}''}$

The inclusion $U_A \hookrightarrow U_{A'}$ gives a long exact seq. in cohomology

$$\cdots \to H^i(U_{A'}) \longrightarrow H^i(U_A) \longrightarrow H^{i+1}(U_{A'}, U_A) \to \cdots$$

Claim: $H^{i+1}(U_{A'}, U_A) \cong H^{i-1}(U_{A''})$

Proof of claim: Let N be a tubular neighborhood of $U_{A''}$ in $U_{A'}$.



Then

$$N \cong \mathbb{C} \times U_{A''}$$

Trivial fiber bundle over $U_{A''}$

and

$$N^\times := N \setminus U_{A''} \cong \mathbb{C}^\times \times U_{A''}$$

Relative Künneth:

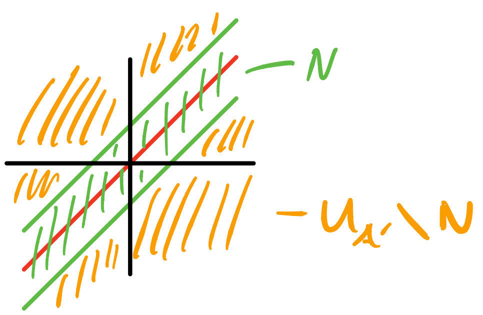
$$H^\bullet(N, N^\times) \cong \underbrace{H^\bullet(\mathbb{C}, \mathbb{C}^\times)}_{H^i = \begin{cases}\mathbb{Z} & i=2 \\ 0 & \text{else}\end{cases}} \otimes H^\bullet(U_{A''})$$

$$\Rightarrow H^{i+1}(N, N^\times) \cong H^{i-1}(U_{A''})$$

(Thom Isomorphism)

On the other hand, $U_A = U_{A'} \setminus U_{A''}$, so

$U_{A'} \setminus N \subseteq U_A$.



- $U_{A'} \setminus (U_{A'} \setminus N) = N$
- $U_A \setminus (U_{A'} \setminus N) = N^\times$

We can excise this subspace

$$H^\bullet(U_{A'}, U_A) \cong H^\bullet(N, N^\times).$$

Next step: Show we have a short exact sequence

$$0 \to OS^i(M\setminus e) \xrightarrow{\text{inclusion}} OS^i(M) \to OS^{i-1}(M/e) \to 0$$

$$x_S \mapsto \begin{cases} x_{S\setminus e} & \text{if } e \in S \\ 0 & \text{otherwise} \end{cases}$$

To show exactness in middle, use the nbc-monomials, which form a basis for $OS^\cdot(M)$.

Once we have this, we can finish the induction.

$$\begin{array}{ccccccccc} 0 & \to & OS^i(M\setminus e) & \to & OS^i(M) & \to & OS^{i-1}(M/e) & \to & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 0 & \to & H^i(U_{A'}) & \to & H^i(U_A) & \to & H^{i-1}(U_{A''}) & \to & 0 \end{array}$$

- the diagram commutes
- the outer maps are iso.s by induction
- exactness on right
- LES $\Rightarrow$ exactness in middle + left
- 5-lemma $\Rightarrow$ iso. in middle