Last week:

Thm (Brieskorn, Orlik-Solomon): Let M be a simple $\mathbb{C}$-representable matroid, and $\mathcal{A}$ a configuration with $M = M(\mathcal{A})$. There is an isomorphism

$$OS^\bullet(M) \longrightarrow H^\bullet(U_{\mathcal{A}})$$
$$x_i \longmapsto \beta_i$$

Def: The Poincaré polynomial of $\mathcal{A}$ is

$$\pi_{\mathcal{A}}(q) = \sum_{i \geq 0} (\operatorname{rank} H^i(U_{\mathcal{A}}))\, q^i$$

Cor: Let M be a simple $\mathbb{C}$-representable matroid and $\mathcal{A}$ a configuration with $M(\mathcal{A}) = M$. Then

$$\begin{aligned}\pi_{\mathcal{A}}(q) &= \sum_{i \geq 0} (\operatorname{rank} OS^i(M))\, q^i \\ &= (-q)^{rk(M)} \chi_M\left(-\tfrac{1}{q}\right) \\ &= \sum_{i=0}^{rk(M)} |w_i|\, q^i.\end{aligned}$$

That is $\operatorname{rank} OS^i(M) = |w_i|$.

Recall: In the proof of the theorem, we found a short exact sequence

$$0 \to OS^i(M\backslash e) \to OS^i(M) \to OS^{i-1}(M/e) \to 0$$

Proof: If $\mathcal{A}$ is the empty arrangement in $\mathbb{C}^0$, so $M = U_{0,0}$, then

$$\pi_{\mathcal{A}}(q) = 1 = (-q)^0 \chi_{U_{0,0}}(-\tfrac{1}{q}).$$

Otherwise, the deletion-contraction s.e.s. implies

$$\pi_{\mathcal{A}}(q) = \pi_{\mathcal{A}'}(q) + q \cdot \pi_{\mathcal{A}''}(q).$$

If e is not a coloop, then by induction

$$\begin{aligned}
\pi_{\mathcal{A}}(q) &= (-q)^{rk(M)} \chi_{M\backslash e}(-\tfrac{1}{q}) + q \cdot (-q)^{rk(M)-1} \chi_{M/e}(-\tfrac{1}{q}) \\
&= (-q)^{rk(M)} \left(\chi_{M\backslash e}(-\tfrac{1}{q}) - \chi_{M/e}(-\tfrac{1}{q}) \right) \\
&= (-q)^{rk(M)} \chi_M(-\tfrac{1}{q}).
\end{aligned}$$

If e is a coloop, then the only difference is $rk(M\setminus e) = rk(M) - 1$. So

$$\pi_{\mathcal{A}}(q) = (-q)^{rk(M)-1} \chi_{M\setminus e}(-\tfrac{1}{q}) + q \cdot (-q)^{rk(M)-1} \chi_{M/e}(-\tfrac{1}{q})$$

$$= \left[(-q)^{rk(M)-1} + q(-q)^{rk(M)-1}\right] \chi_{M\setminus e}(-\tfrac{1}{q})$$

$$= (-q)^{rk(M)} \left(-\frac{1}{q} - 1\right) \chi_{M\setminus e}(-\tfrac{1}{q})$$

$$= (-q)^{rk(M)} \chi_M(-\tfrac{1}{q}).$$

∎

Final remarks on OS-algebras:

- This corollary tells us we can think of $OS^\bullet(M)$ as a "categorification" of χ_M.
- Although $H^\bullet(U_{\mathcal{A}})$ is only defined for $\mathbb{C}$-representable matroids, but $OS^\bullet(M)$ is defined for all matroids. So we can attach a "cohomology ring of the complement" to any matroid.
- We might hope that this relationship helps us prove log-concavity of the $|w_i|$. It doesn't. But something similar will.