Compactification of $U_{\mathcal{A}}$

Continue to let

- $\mathcal{A} = \{v_e \mid e \in E\}$ a configuration in a $\mathbb{C}$-vector space V (WLOG $\mathcal{A}$ spans V), with $M(\mathcal{A}) = M$ a simple matroid.
- $\{H_e \mid e \in E\}$ the associated hyperplane arrangement in V^*.
- $U_{\mathcal{A}} := V^* \setminus \bigcup_{e \in E} H_e$

We can projectivize this picture to get

- $\mathbb{P}\mathcal{A}$, a configuration of points in $\mathbb{P}V$
- An arrangement of projective hyperplanes in $\mathbb{P}V^*$
- The complement of this arrangement, which is $\mathbb{P}U_{\mathcal{A}}$.

We've seen that $H^\bullet(U_{\mathcal{A}}) \cong OS^\bullet(M)$.

<u>Nice</u>: $\operatorname{rank} H^i(U_{\mathcal{A}}) = |w_i|$

<u>Not nice</u>: $U_{\mathcal{A}}$ is not compact

<u>One improvement</u>: Replace U_A with $\mathbb{P}U_A$.

Then $H^\bullet(\mathbb{P}U_A) \cong \overline{OS}^\bullet(M)$, where

$\overline{OS}^\bullet(M)$ = subring of $OS^\bullet(M)$ gen. by differences $x_i - x_j$.

<u>Nice</u>: rank $\overline{OS}^i(M) = \mu^i$ (coeff. of q^{r-1-i} in $\overline{\chi}_M$)

<u>Not nice</u>: $\mathbb{P}U_A$ still not compact.

We'd like to find a <u>compactification</u>, a compact space (complex manifold) containing $\mathbb{P}U_A$ as a dense open subset.

<u>Easy but unsubtle choice</u>: The one-point compactification.

<u>Another unsubtle choice</u>: $\mathbb{P}V^* \supset \mathbb{P}U_A$

We'll turn to algebraic geometry to find a better compactification.

Suppose X is a smooth complex variety (an "algebraic" complex manifold).

If $\overline{X}$ is a compactification (Bad notation!) then what does the boundary divisor

$$\partial \overline{X} := \overline{X} \setminus X$$

look like?

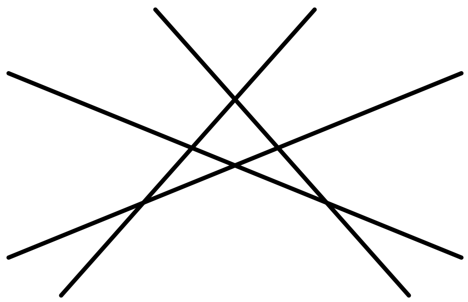
How did we "fix" non-compactness?

Thm (Nagata 1962): We can always find a compactification $\overline{X}$ such that the boundary $\partial \overline{X}$ is

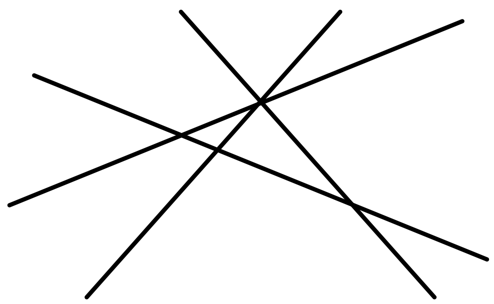
- a union of smooth varieties
- these smooth components intersect transversally;

i.e. $\partial \overline{X}$ is a simple normal crossings (SNC) divisor.

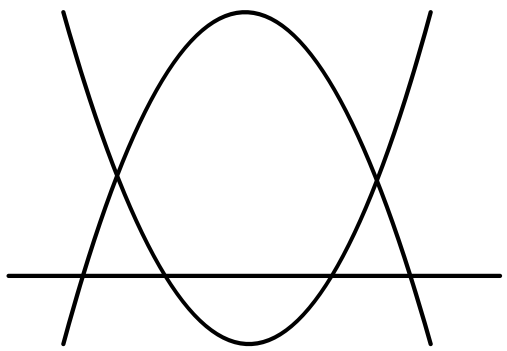
Intuitively, an SNC divisor locally looks like the intersection of coordinate hyperplanes.

Ex:  is an SNC divisor in $\mathbb{P}^2$

$M = U_{3,4}$

Ex:  is not an SNC divisor in $\mathbb{P}^2$

$M = U_{1,1} \oplus U_{2,3}$

Ex:  is an SNC divisor in $\mathbb{P}^2$

In the case of $X = \mathbb{P}U_d$, De Concini and Procesi ('95) give an explicit constuction of an SNC compactification, called the <u>wonderful compactification</u> of $\mathbb{P}U_d$.