Recall: • M a simple $\mathbb{C}$-representable matroid
• $\mathcal{A}$ a configuration in V with $M = M(\mathcal{A})$
• $U_{\mathcal{A}}$ the hyperplane arrangement complement in V^*

Goal: Describe De Concini and Procesi's <u>wonderful compactification</u> of $\mathbb{P}U_{\mathcal{A}}$.

Notation: For each flat F of M, set

$$H_F := \bigcap_{e \in F} H_e \subseteq V^*$$

Note: • $H_\emptyset = V^*$
• $H_E = \{0\}$
• $H_G \subseteq H_F \iff G \supseteq F$

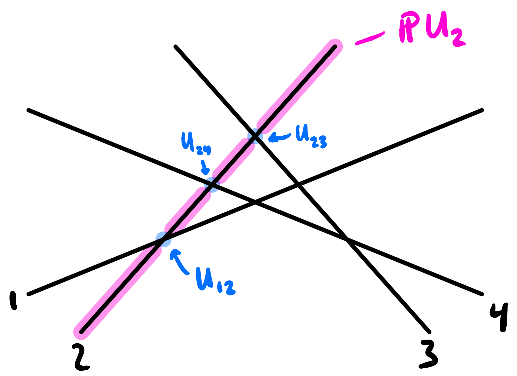
and

$$U_F := H_F \setminus \left(\bigcup_{G \supsetneq F} H_G \right)$$

Note: • $\overline{U_F} = H_F$
• $U_\emptyset = U_{\mathcal{A}}$
• $U_E = \{0\}$

Then
$$V^* = \coprod_{F\in\mathcal{L}(M)} U_F$$
and
$$\mathbb{P}V^* = \coprod_{\substack{F\in\mathcal{L}(M)\\ F\neq E}} \mathbb{P}U_F$$

Ex:



in $\mathbb{P}^2$ $(M = U_{3,4})$

The composition
$$\underset{U_\emptyset}{\overset{\|}{U_A}} \hookrightarrow V^* \twoheadrightarrow V^*/H_F$$

A vector space of dimension $r - \mathrm{crk}(F) = \mathrm{rk}(F)$

gives
$$\mathbb{P}U_A \longrightarrow \mathbb{P}(V^*/H_F)$$
for each non-empty flat F.

Putting these together, we get

$$\psi : \mathbb{P}U_{\mathcal{A}} \longrightarrow \prod_{\emptyset \neq F \in \mathcal{L}(M)} \mathbb{P}(V^*/H_F)$$

For each $x \in \mathbb{P}U_{\mathcal{A}}$, $\psi(x)$ records the "direction" from x to H_F

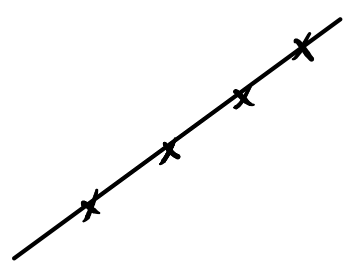
<u>Observation</u>: ψ is an open embedding (in the coordinate indexed by $F = E$, it's $\mathbb{P}U_{\mathcal{A}} \hookrightarrow \mathbb{P}V^*$), so the image $\psi(\mathbb{P}U_{\mathcal{A}})$ is an open subvariety.

<u>Def</u>: The closure of the image $\psi(\mathbb{P}U_{\mathcal{A}})$ is called the <u>wonderful compactification</u> of $\mathbb{P}U_{\mathcal{A}}$ (also the <u>wonderful model</u> of $\{H_e\}$), and is denoted $Y_{\mathcal{A}}$.

In rank 2, this is boring.

<u>Ex</u>: $\mathbb{P}U_{\mathcal{A}} = \mathbb{P}^1 \setminus \{n \text{ distinct points}\}$

$M = U_{2,n}$



Then

$$\psi : \mathbb{P}^1 \setminus \{n \text{ pts}\} \longrightarrow \mathbb{P}^1 \times \cancel{\underbrace{\mathbb{P}^0 \times \mathbb{P}^0 \times \cdots \times \mathbb{P}^0}_{n}}$$

$$[x:y] \longmapsto [x:y]$$

i.e. ψ is the inclusion of $\mathbb{P}^1 \setminus \{n \text{ pts}\}$ into $\mathbb{P}^1$.

So its closure is

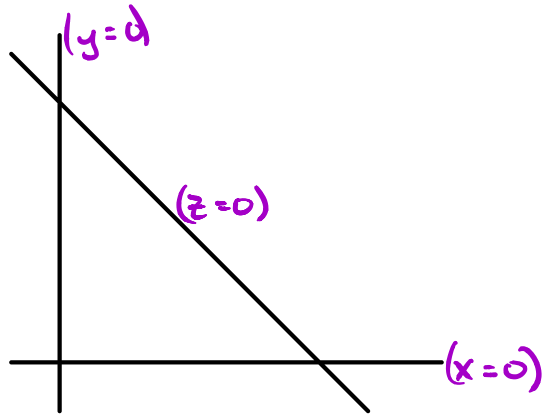
$$Y_A = \overline{\mathbb{P} U_A} = \mathbb{P}^1.$$

It gets more interesting in rank 3.

<u>Ex</u>: Let $A = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$, so $M = U_{3,3}$.

The hyperplane arr. is the 3 coordinate planes in $V^* \cong \mathbb{C}^3$.

In $\mathbb{P}V^* \cong \mathbb{P}^2$:



$$\mathbb{P} U_A = \{[x:y:z] \in \mathbb{P}^2 \mid x, y, z \text{ all nonzero}\}$$

$$\cong (\mathbb{C}^\times)^3 / \mathbb{C}^\times \cong (\mathbb{C}^\times)^2$$

Then

$$\psi : \mathbb{P}U_A \longrightarrow \mathbb{P}^2 \times \mathbb{P}(\mathbb{C}^3/_{z\text{-axis}}) \times \mathbb{P}(\mathbb{C}^3/_{y\text{-axis}}) \times \mathbb{P}(\mathbb{C}^3/_{x\text{-axis}}) \cancel{\times \mathbb{P}^0 \times \mathbb{P}^0 \times \mathbb{P}^0}$$

$$[x:y:z] \longmapsto \left([x:y:z], [x:y], [x:z], [y:z], \cancel{[x], [y], [z]}\right)$$

A point in the image is totally determined by the 1st coord. But what do we add in the closure?

If x and y are both nonzero, then we can approach $[x:y:0]$ along a path in $\mathbb{P}U_A \cong \psi(\mathbb{P}U_A)$. In the limit, we get

$$([x:y:0], [x:y], [x:0], [y:0])$$

$$= ([x:y:0], [x:y], [1:0], [1:0])$$

So we can recover any point in the hyperplane $(z=0)$, other than $[1:0:0]$ and $[0:1:0]$.

Similar for $(y=0)$ and $(z=0)$.

What happens if we approach $[1:0:0]$ along a path in U_A?

The tangent line of the path at $[1:0:0]$ is defined by

$$\overset{0}{\cancel{a}}x + by + cz = 0$$

$$by + cz = 0 \to z = -\frac{b}{c}y$$

for some $b, c \in \mathbb{C}$ not both zero.

$$[y:z] = [y : -\tfrac{b}{c}y] = [c:-b]$$

So in the limit, we get

$$([1:0:0], [1:0], [1:0], [c:-b])$$

So we get a copy of $\mathbb{P}^1$ lying "above" $[1:0:0]$, corresponding to the directions from which can approach it.

Similarly for $[0:1:0]$ and $[0:0:1]$.

So Y_A compactifies $\mathbb{P}U_A$ by adding the bounding divisor

$\partial Y_A =$