Last Week: The wonderful compactification

- $\mathcal{A}$ a configuration in $\mathbb{C}$-vector space V
- $U_{\mathcal{A}}$ the complement of the associated hyperplane arrangement in V^*

We have an open embedding

$$\psi : \mathbb{P}U_{\mathcal{A}} \longrightarrow \prod_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset}} \mathbb{P}(V^*/H_F)$$

The wonderful compactification $Y_{\mathcal{A}}$ is the closure of the image $\psi(\mathbb{P}U_{\mathcal{A}})$.

Key observation: The projection onto $\mathbb{P}(V^*/\underset{\{0\}}{\underset{\parallel}{H_E}}) = \mathbb{P}V^*$ gives a surjection

$$\pi : Y_{\mathcal{A}} \longrightarrow \mathbb{P}V^*$$

which restricts to an isomorphism over $\mathbb{P}U_{\mathcal{A}}$:

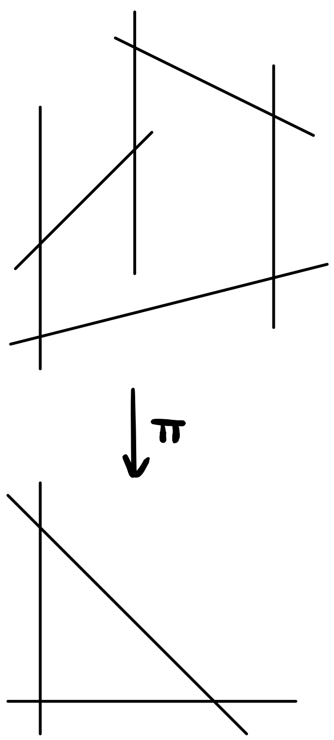
$$\pi^{-1}(\mathbb{P}U_{\mathcal{A}}) \xrightarrow{\sim} \mathbb{P}U_{\mathcal{A}}$$

The complement of $\pi^{-1}(\mathbb{P}\mathcal{U}_\mathcal{A})$ in $Y_\mathcal{A}$ is the <u>boundary divisor</u> $\partial Y_\mathcal{A}$. By construction, π maps $\partial Y_\mathcal{A}$ surjectively onto $\bigcup_{e \in E} H_e$.

What does this look like?

<u>Ex</u>: For the arrangement of coordinate lines in $\mathbb{P}^2$ (a representation of $U_{3,3}$), we found the boundary divisor to be

$\partial Y_\mathcal{A} =$



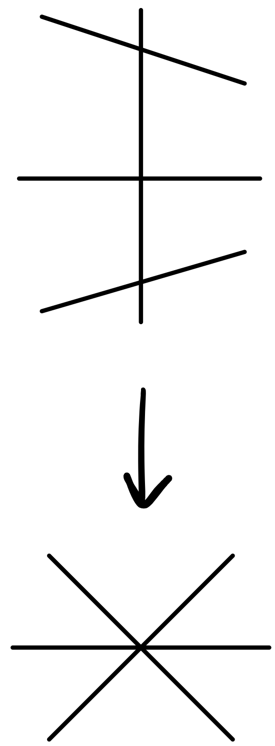
<u>Notes</u>

- $Y_\mathcal{A} \setminus \partial Y_\mathcal{A} \cong \mathbb{P}\mathcal{U}_\mathcal{A}$
- The "horizontal" components don't intersect.
- For each flat F $(\neq \emptyset, E)$, $\pi^{-1}(\mathbb{P}H_F)$ is a union of components, while $\pi^{-1}(\mathbb{P}\mathcal{U}_F)$ is a single component.

This example is kind of silly, because we already had an SNC compactification of $\mathbb{P}\mathcal{U}_\mathcal{A}$, but it illustrates the basic features.

In rank 3, we replace each intersection point (rank 2 flats) with a "vertical" copy of $\mathbb{P}^1$:

<u>Local picture</u>:



This is a <u>blowup</u>, and in general Y_A can be defined as a sequence of blowups.

In higher rank, the picture is more complicated (and impossible to draw), but the intuition here mostly carries over.

Boundary Components

Def: For each proper non-empty flat F, let

$$D_F := \overline{\pi^{-1}(\mathbb{P}U_F)} \subseteq Y_A$$

Observe:

- $\partial Y_A = \bigcup_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset, E}} D_F$

- $\pi^{-1}(H_e) = \bigcup_{F \ni e} D_F$

Thm: Let F be a proper, non-empty flat. Then

$$D_F \cong \mathbb{P}(H_F) \times \mathbb{P}(V^*/H_F)$$

Cor:
$$\begin{aligned} \dim D_F &= (r - rk(F) - 1) + (r - (r - rk(F)) - 1) \\ &= r - 2 \\ &= \dim \mathbb{P}U_A - 1 \\ &= \dim Y_A - 1 \end{aligned}$$

Cor: $D_F \cap D_G \neq \emptyset \iff F \subseteq G$ or $G \subseteq F$