<u>Recall</u>: • $\mathcal{A}$ a configuration in $\mathbb{C}$-vector space V

• $U_{\mathcal{A}}$ the complement of the associated hyperplane arrangement in V^*

Then the <u>wonderful compactification</u> $Y_{\mathcal{A}}$ is the closure of the image of

$$\psi : \mathbb{P}U_{\mathcal{A}} \hookrightarrow \prod_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset}} \mathbb{P}(V^*/H_F),$$

where $H_F = \bigcap_{e \in F} H_e$.

For a proper, non-empty flat F, we set

$$D_F := \overline{\pi^{-1}(\mathbb{P}U_F)}$$

where $U_F = H_F \setminus \bigcup_{G \supsetneq F} H_G$ and

$$\pi : Y_{\mathcal{A}} \longrightarrow \mathbb{P}V^*$$

is the natural projection.

Thm: For each proper, non-empty flat F,

$$D_F \cong \underbrace{\mathbb{P}(H_F)}_{\operatorname{crk} F - 1} \times \underbrace{\mathbb{P}(V^*/H_F)}_{(r - \operatorname{crk} F) - 1}$$

Cor: $\dim_{\mathbb{C}} D_F = \operatorname{rk} M - 2$

$= \dim_{\mathbb{C}} Y_A - 1$

Proof sketch: Approach $\mathbb{P}U_F$ along a path in $\mathbb{P}U_A$.

In the limit, we get a point $x \in \mathbb{P}U_F \subseteq \mathbb{P}V^*$.

Via

$$\underset{\substack{\parallel \\ H_F \setminus \bigcup_{G \supsetneq F} H_G}}{U_F} \hookrightarrow V^* \twoheadrightarrow V^*/H_G$$

x will have a well-defined image in $\mathbb{P}(V^*/H_G)$

so long as $H_F \not\subseteq H_G \iff G \not\subseteq F$.

By continuity, whenever this image is defined, it will be a coordinate of the limit point in Y_A.

For the remaining coordinates, we look at the tangent line to the path at x.
This determines a point

$$u \in \mathbb{P}(V^*/H_F)$$

which will be a coordinate of the limit point.

By continuity, the remaining coordinates are given by the images of u under the maps

$$\mathbb{P}(V^*/H_F) \twoheadrightarrow \mathbb{P}(V^*/H_G)$$

which are well-defined whenever $H_F \subseteq H_G$

$$\iff G \subseteq F.$$

Thus, the limit point in $Y_{\mathcal{A}}$ is uniquely determined by $x \in \mathbb{P}U_F$ and $u \in \mathbb{P}(V^*/H_F)$.
In the closure, x can be any point in $\mathbb{P}H_F$, so

$$D_F \cong \mathbb{P}H_F \times \mathbb{P}(V^*/H_F).$$

Cor: $D_F \cap D_G \neq \emptyset \iff F \subseteq G$ or $G \subseteq F$

"F and G are comparable"

The Chow ring

If $M = M(\mathcal{A})$ is a simple $\mathbb{C}$-representable matroid, and $Y_\mathcal{A}$ is the wonderful compactification.

We know

- $\partial Y_\mathcal{A} = Y_\mathcal{A} \setminus \pi^{-1}(\mathbb{P}U_\mathcal{A}) = \bigcup_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset, E}} D_F$

- D_F has complex codimension one in $Y_\mathcal{A}$

- $D_F \cap D_G \neq \emptyset \iff F \subseteq G$ or $G \subseteq F$

- $\pi^{-1}(H_e) = \bigcup_{F \ni e} D_F$

Each D_F is a subvariety (closed submanifold) so defines a class $[D_F] \in H^2(Y_\mathcal{A}; \mathbb{Z})$.

Thm (De Concini–Procesi '95): Let M be a simple $\mathbb{C}$-representable matroid on ground set E, and let $\mathcal{A}$ be a configuration in a $\mathbb{C}$-vector space V representing M.

Then the map

$$\mathbb{Z}[x_F \mid F \in \mathcal{L}(M) \setminus \{E, \emptyset\}] \longrightarrow H^\bullet(Y_{\mathcal{A}}; \mathbb{Z})$$
$$x_F \longmapsto [D_F]$$

is surjective, i.e., the $[D_F]$ generate $H^\bullet(Y_{\mathcal{A}}; \mathbb{Z})$.

The kernel is generated by

- $x_F x_G$ for F, G incomparable

- $\sum_{F \ni e} x_F - \sum_{G \ni f} x_G$ for $e, f \in E$.