Last time:

Thm (DeConcini - Procesi '95): Let M be a simple $\mathbb{C}$-representable matroid on ground set E, and let $\mathcal{A}$ be a configuration in a $\mathbb{C}$-vector space V s.t. $M = M(\mathcal{A})$.

Then the map

$$\mathbb{Z}[x_F \mid F \in \mathcal{L}(M) \setminus \{\emptyset, E\}] \longrightarrow H^\bullet(Y_{\mathcal{A}}; \mathbb{Z})$$

$$x_F \longmapsto [D_F]$$

is a surjection.

The kernel is generated by

- $x_F x_G$; $\quad F, G$ incomparable
- $\sum_{F \ni e} x_F - \sum_{G \ni f} x_G$; $\quad e, f \in E$

In general, the Chow ring of a smooth projective (compact) complex variety X, is the subring

$$A^\bullet(X) \subseteq H^{2\bullet}(X)$$

generated by classes of algebraic subvarieties.

Cor: $A^\bullet(Y_{\mathcal{A}}) = H^{2\bullet}(Y_{\mathcal{A}})$

Def: Let M be a simple matroid. The (integral) Chow ring of M is

$$A^\bullet(M) := \frac{\mathbb{Z}[x_F \mid F \in \mathcal{L}(M) \setminus \{\emptyset, E\}]}{I_M + J_M}$$

where

$$I_M = (x_F x_G \mid F, G \text{ incomparable})$$

$$J_M = \Big(\sum_{F \ni e} x_F - \sum_{G \ni f} x_G \;\Big|\; e, f \in E\Big)$$

and each x_F has degree 1.

$\underline{\text{Ex}}$: $A^\bullet(U_{3,4})$ has 10 generators

$$x_1, \dots, x_4 \quad \text{and} \quad x_{12}, \dots, x_{34} \quad \in A^1(U_{3,4})$$

The linear relations tell us the 4 elements

$$x_1 + x_{12} + x_{13} + x_{14}$$
$$x_2 + x_{12} + x_{23} + x_{24}$$
$$x_3 + x_{13} + x_{23} + x_{34}$$
$$x_4 + x_{14} + x_{24} + x_{34}$$

are equal in $A^\bullet(U_{3,4})$.

$\Rightarrow$ rank $A^1(U_{3,4}) = 7$

The quadratic relations tell us that every product of two generators is zero except for possibly

$$x_i x_{ij}, \qquad x_i^2, \qquad x_{ij}^2.$$

$\underline{\text{Exercise 2}}$: For all i,j

- $x_i x_{ij} = x_1 x_{12}$
- $x_i^2 = -2x_1 x_{12}$
- $x_{ij}^2 = -x_1 x_{12}$.

$\Rightarrow A^2(U_{3,4})$ gen. by $x_1 x_{12}$

$\Rightarrow A^2(U_{3,4}) \cong \mathbb{Z}$ or $\{0\}$.

$\Rightarrow A^k(U_{3,4}) = \{0\}$ for $k \geq 3$.