<u>Recall</u>: The <u>Chow ring</u> of M is

$$A^\bullet(M) := \frac{\mathbb{Z}[x_F \mid F \in \mathcal{L}(M) \setminus \{\emptyset, E\}]}{I_M + J_M},$$

where

$$I_M = (x_F x_G \mid F, G \text{ incomparable}),$$

$$J_M = \left(\sum_{F \ni e} x_F - \sum_{G \ni f} x_G \;\middle|\; e, f \in E\right)$$

and each x_F has degree 1.

<u>Def</u>: Let M be a simple matroid of rank r on ground set E. Let $i \in E$, and define

$$\alpha = \sum_{F \ni i} x_F, \qquad \beta = \sum_{F \not\ni i} x_F \quad \in A^1(M)$$

<u>Observation</u>: α and β are independent of the choice of $i \in E$.

- $\sum_{F \ni i} x_F = \sum_{G \ni j} x_G$ by the linear relations J_M

- $\beta = \sum_{F} x_F - \alpha$

Prop: Let $\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq E$ be a flag.

① If the flag is initial ($\mathrm{rk}\, F_i = i$ for all i), then

$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha^{r-1-k} = \alpha^{r-1} \in A^{r-1}(M)$$

② If the flag is not initial, then

$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha^{r-1-k} = 0 \qquad \in A^{r-1}(M)$$

Proof: We begin with an observation. If F is a (non-empty, proper) flat and $i \notin F$, then

★ $$x_F \alpha = x_F \sum_{G \ni i} x_G = x_F \sum_{\substack{G \ni i \\ \text{s.t. } F \subsetneq G}} x_G$$

★ In particular, if $\mathrm{rk}\, F = r-1$, then $x_F \alpha = 0$

Proof of (2): For the flag to not be initial, we must have $k < r-1$

If $k = r-2$, then $\mathrm{rk}(F_{r-2}) = r-1$, so the product

$$x_{F_1} \cdots x_{F_{r-2}} \cdot \alpha = 0$$

by ★.

For general k, choose some $i \notin F_k$.

By ★,

$$x_{F_1} \cdots x_{F_k} \cdot \alpha^{r-1-k} = x_{F_1} \cdots x_{F_k} \left(\sum_{\substack{G \ni i \\ \text{s.t.} \\ F_k \subseteq G}} x_G \right) \alpha^{r-1-k-1}.$$

But this is a sum over length-$(k+1)$ flags which are not initial. So it is zero by induction. ✓

Proof of ①: Induct on k.

If $k=1$, then $\emptyset \subsetneq F_1 \subsetneq E$ where $F_1 = \{i\}$.

So
$$\alpha^{r-1} = \Big(\sum_{G \ni i} x_G\Big)\, \alpha^{r-2}$$

By ②, $x_G \alpha^{r-2} = 0$ unless $\emptyset \subsetneq G \subsetneq E$ is initial, i.e. $\operatorname{rk} G = 1$. But the only such flat is $G = \{i\} = F_1$.

So
$$\alpha^{r-1} = x_{F_1} \alpha^{r-2}.$$

If $k>1$, the first $k-1$ steps of a k-step initial flag
$$\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq F_k \subsetneq E$$
will be initial also.

By induction
$$\alpha^{r-1} = x_{F_1} \cdots x_{F_{k-1}}\, \alpha^{r-1-(k-1)}.$$

Choose $i \in F_k \setminus F_{k-1}$. By $\star$,
$$\alpha^{r-1} = x_{F_1} \cdots x_{F_{k-1}} \Big(\sum_{\substack{G \ni i \\ G \supseteq F_{k-1}}} x_G\Big)\, \alpha^{r-1-k}.$$

By ② each term corresponding to a non-initial flag

$$\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq G \subsetneq E$$

is zero.

So we need $G \supset F_{k-1}$. But the only such G is

$$G = \mathrm{cl}\left(F_{k-1} \cup \{i\}\right) = F_k .$$

So

$$\alpha^{r-1} = x_{F_1} \cdots x_{F_k} \alpha^{r-1-k} .$$

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