Last time:

$$\alpha = \sum_{F \ni i} x_F, \quad \beta = \sum_{F \not\ni i} x_F \quad \in A^1(M),$$

where i is any element in the ground set of M.

Then

Prop: Let $\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq E$ be a flag.

① If the flag is initial ($\operatorname{rk} F_i = i$ for all i), then

$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha^{r-1-k} = \alpha^{r-1} \in A^{r-1}(M)$$

② If the flag is not initial, then

$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha^{r-1-k} = 0 \in A^{r-1}(M)$$

Thm: Let M be a simple matroid of rank r on ground set E. Then

(1) For each $k \geq 0$, $A^k(M)$ is generated by square-free monomials $x_{F_1} x_{F_2} \cdots x_{F_k}$, where

$$\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq E.$$

(2) $A^k(M) = 0$ for $k \geq r$.

(3) $A^{r-1}(M) \cong \mathbb{Z}$ is generated by α^{r-1}.

Proof: (1) is an exercise (last time).

(2) follows from (1).

(3) is surprisingly difficult.

- (1) together with the previous proposition shows
$$A^{r-1}(M) = \mathbb{Z}\alpha^{r-1}$$

- It remains to show $\alpha^{r-1} \neq 0$. 2 basic approaches:
  - (Feichtner - Yuzvinsky) Find Gröbner basis of the ideal of relations for $A^\bullet(M)$ and use it to find a basis of $A^\bullet(M)$ containing α^{r-1}.
  - (Adiprasito - Huh - Katz) Show that $A^\bullet(M)$ can be identified with certain functions on a unimodular fan and α^{r-1} corresponds to a nonzero function.

$\blacksquare$

Def: The <u>degree map</u> of M is the isomorphism

$$\begin{aligned} \deg : A^{r-1}(M) &\xrightarrow{\sim} \mathbb{Z} \\ \alpha^{r-1} &\longmapsto 1 \end{aligned}$$

Note: By the proposition, if $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq E)$ is any complete flag, then

$$x_{\mathcal{F}} := x_{F_1} x_{F_2} \cdots x_{F_{r-1}} = \alpha^{r-1}$$

$$\Rightarrow \deg(x_{\mathcal{F}}) = \deg(\alpha^{r-1}) = 1.$$

Now, choose a total ordering of E (i.e., a bijection $E \cong [n]$).

Recall that a flag

$$\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq [n]$$

is <u>descending</u> if

$$\min(F_1) > \min(F_2) > \cdots > \min(F_k) > 1$$

Prop: For each $0 \leq k \leq r$,

$$\beta^k = \sum_{\substack{\mathcal{F} \text{ descending} \\ k\text{-step} \\ \text{flag}}} x_{\mathcal{F}}$$

Proof: A 1-step flag $\emptyset \subsetneq F \subsetneq [n]$ is descending if and only if $1 \notin F$.

By definition,

$$\beta = \sum_{F \not\ni 1} x_F$$

So the theorem holds for $k=1$.

For the induction step,

$$\beta^{k+1} = \beta \cdot \beta^k = \sum_{\substack{\mathcal{F} \text{ descending} \\ k\text{-step flag}}} \beta x_{\mathcal{F}}$$

For each descending k-step flag $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq [n])$ let $i_{\mathcal{F}} = \min(F_1)$. Then

$$\beta x_{\mathcal{F}} = \left(\sum_{G \not\ni i_{\mathcal{F}}} x_G\right) x_{\mathcal{F}} = \sum_{\substack{G \not\ni i_{\mathcal{F}} \\ G \subseteq F_1}} x_G x_{\mathcal{F}},$$

i.e. this is a sum over all $(k+1)$-step flags of the form

$$\emptyset \subsetneq G \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq [n]$$

where

$$\min(G) > i_{\mathcal{F}} = \min(F_1).$$

$\blacksquare$

Cor: For $0 \leq k \leq r-1$,

$$\deg(\alpha^{r-1-k}\beta^k) = \mu^k,$$

where $(-1)^k\mu^k$ is the coefficient of q^{r-1-k} in $\overline{\chi}_M(q)$.

Recall: $\mu^k = \#\{\text{initial, descending } k\text{-step flags}\}$.

Proof: $$\alpha^{r-1-k}\beta^k = \sum_{\substack{\mathcal{F}\ k\text{-step} \\ \text{descending} \\ \text{flag}}} x_{\mathcal{F}}\,\alpha^{r-1-k} = \sum_{\substack{\mathcal{F}\ k\text{-step} \\ \text{initial, descending} \\ \text{flag}}} \alpha^{r-1}$$

$$= \mu^k \alpha^{r-1}.$$

$\blacksquare$