# The Kähler package

The key to Adiprasito-Huh-Katz's work is to show $A^\bullet(M)$ "behaves like" the Chow ring of a smooth projective $\mathbb{C}$-variety, even when $M$ is not $\mathbb{C}$-representable.

Specifically, $A^\bullet(M)$ satisfies 3 properties, which together form the <u>Kähler package</u>.

Let $M$ be a rank $r$ simple matroid on ground set $E$.

We let $A^\bullet(M)_{\mathbb{R}} := A^\bullet(M) \otimes_{\mathbb{Z}} \mathbb{R}$.

<u>Def</u>: A function $c: 2^E \to \mathbb{R}$ is <u>strictly submodular</u> if

- $c(\emptyset) = c(E) = 0$

and

- If $A, B \subseteq E$ are incomparable, then

$$c(A \cup B) + c(A \cap B) < c(A) + c(B).$$

**Ex:** $c(A) = |A| \cdot |E \setminus A|$ is strictly submodular.

**Def:** If $c : 2^E \to \mathbb{R}$ is strictly submodular, let

$$\ell_c = \sum_F c(F)\, x_F \quad \in A^1(M)_{\mathbb{R}}.$$

We call $\ell_c$ an <u>ample class</u>, and we let

$$K(M) = \{ \ell_c \mid c \text{ strictly submodular} \} \subseteq A^1(M)$$

be the <u>ample cone</u> of $M$.

**Def:** The <u>nef cone</u> $N(M)$ is the closure of $K(M)$ in $A^1(M)_{\mathbb{R}}$. Elements of $N(M)$ are called <u>nef</u> classes.

**Lemma:** The elements $\alpha, \beta \in A^1(M)$ are nef.

**Proof:** Let $i \in E$ and define $\delta_i : 2^E \to \mathbb{R}$ by

$$\delta_i(A) = \begin{cases} 0 & i \notin A \text{ or } A = E \\ 1 & i \in A \subsetneq E \end{cases}.$$

So $\alpha = \sum_F \delta_i(F)\, x_F$.

Let $c: 2^E \to \mathbb{R}$ be a strictly submodular function. Then for any $\epsilon > 0$, the function

$$\delta_i + \epsilon \cdot c$$

is strictly submodular: For incomparable $A, B \subseteq E$

$$c(A \cup B) + c(A \cap B) < c(A) + c(B)$$

$$\Rightarrow \epsilon \cdot c(A \cup B) + \epsilon \cdot c(A \cap B) < \epsilon \cdot c(A) + \epsilon \cdot c(B)$$

- $i \notin A$, $i \notin B$ $\Rightarrow$ $\delta_i$ makes no contribution ✓
- $i \in A$, $i \notin B$ or $i \notin A$, $i \in B$ $\Rightarrow$ $\delta_i$ contributes 1 to right side, at most 1 to left side ✓
- $i \in A$, $i \in B$ $\Rightarrow$ $\delta_i$ contributes 2 to right side, at most 2 to left sides ✓

So

$$\sum_F \left( \delta_i(F) + \epsilon \cdot c(F) \right) x_F = \alpha + \epsilon\, \ell_c$$

is ample. In the limit as $\epsilon \to 0$, we get $\alpha \in \mathcal{N}(M)$.

Similarly, $\beta \in \mathcal{N}(M)$.

## Thm (Adiprasito-Huh-Katz '15)

Let $M$ be a simple matroid of rank $r$.
Let $0 \le k \le \frac{r-1}{2}$ and $\ell \in K(M)$.

① <u>Poincare duality</u>:

$$A^k(M) \times A^{r-1-k}(M) \longrightarrow \mathbb{Z}$$
$$(\eta_1, \eta_2) \longmapsto \deg(\eta_1 \cdot \eta_2)$$

is a perfect pairing.

$A^k(M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(A^{r-1-k}(M), \mathbb{Z})$

② <u>Hard Lefschetz</u>:

$$L_\ell^k : A^k(M)_{\mathbb{R}} \longrightarrow A^{r-1-k}(M)_{\mathbb{R}}$$
$$\eta \longmapsto \ell^{r-1-2k} \cdot \eta$$

is an isomorphism of $\mathbb{R}$-vector spaces.

## ③ Hodge-Riemann Relations:

The symmetric bilinear form

$$Q_\ell^k : A^k(M)_{\mathbb{R}} \times A^k(M)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

$$(\eta_1, \eta_2) \longmapsto (-1)^k \deg(\ell^{r-1-2k} \cdot \eta_1 \cdot \eta_2)$$

is positive definite on the primitive cohomology

$$P_\ell^k := \ker \ell \cdot L_\ell^k$$

$$= \{\eta \in A^k(M)_{\mathbb{R}} \mid \underbrace{\ell \cdot L_\ell^k(\eta)}_{= \ell^{r-2k} \cdot \eta} = 0\}$$

### Notes on the proof:

- Long and technical.
- The key insight is that the Kähler package is preserved under a "matroidal flip."
- Somehow, it is most natural to prove Hard Lefschetz and Hodge-Riemann simultaneously

# What is this good for?

Hard Lefschetz (HL) implies that, for $k \leq \frac{r-1}{2}$,

$$\begin{aligned} A^{k-1}(M)_{\mathbb{R}} &\longrightarrow A^k(M)_{\mathbb{R}} \\ \eta &\longmapsto \ell \cdot \eta \end{aligned}$$

is injective.

Thus,

$$A^k(M)_{\mathbb{R}} = P_\ell^k \oplus \ell \cdot A^{k-1}(M)_{\mathbb{R}}.$$

So we get the "Lefschetz decomposition"

$$A^k(M)_{\mathbb{R}} = P_\ell^k \oplus \ell \cdot P_\ell^{k-1} \oplus \ell^2 P_\ell^{k-2} \oplus \cdots \oplus \ell^k P_\ell^0$$

This decomposition is orthogonal with respect to the Hodge-Riemann form $Q_\ell^k$:

If $\underbrace{\eta_1 \in P_\ell^{k-i}}_{\ell^{r-2(k-i)} \cdot \eta_1 = 0}$, $\eta_2 \in P_\ell^{k-j}$, where $i < j \leq k$,

then

$$Q_\ell^k(\ell^i \eta_1, \ell^j \eta_2)$$

$$= (-1)^k \deg\left(\ell^{r-1-2k} \cdot \ell^i \eta_1 \cdot \ell^j \eta_2\right)$$

$$= (-1)^k \deg\left(\ell^{j-i-1} \cdot \underbrace{\ell^{r-2(k-i)} \eta_1}_{=0} \cdot \eta_2\right)$$

$$= 0.$$