Last week

Thm (Adiprasito–Huh–Katz '15)

M simple matroid of rank r, $0 \le k \le \frac{r-1}{2}$, $\ell \in K(M)$.

(PD) $$A^k(M) \times A^{r-1-k}(M) \to \mathbb{Z}$$
$$(\eta_1, \eta_2) \longmapsto \deg(\eta_1 \cdot \eta_2)$$

is a perfect pairing

(HL) $$L_\ell^k : A^k(M)_\mathbb{R} \longrightarrow A^{r-1-k}(M)_\mathbb{R}$$
$$\eta \longmapsto \ell^{r-1-2k} \cdot \eta$$

is an isomorphism of $\mathbb{R}$-vector spaces.

(HR) $$Q_\ell^k : A^k(M)_\mathbb{R} \times A^k(M)_\mathbb{R} \longrightarrow \mathbb{R}$$
$$(\eta_1, \eta_2) \longmapsto (-1)^k \deg(\ell^{r-1-2k} \cdot \eta_1 \cdot \eta_2)$$

is positive definite on $P_\ell^k := \ker \underbrace{\ell \cdot L_\ell^k}_{= \text{mult. by } \ell^{r-2k}}$.

Now, (HL) implies

$$A^k(M)_{\mathbb{R}} = P_\ell^k \oplus \ell \cdot A^{k-1}(M)_{\mathbb{R}}.$$

for $k \leq \frac{r-1}{2}$.

<u>Proof</u>: Multiplication by ℓ gives an injection

$$A^{k-1}(M)_{\mathbb{R}} \xrightarrow{\cdot \ell} A^k(M)_{\mathbb{R}}.$$

So $\ell \cdot A^{k-1}(M)_{\mathbb{R}} \subseteq A^k(M)_{\mathbb{R}}$.

Now, let $\eta \in A^k(M)_{\mathbb{R}}$. Then

$$\ell \cdot L_\ell^k(\eta) = \ell^{r-2k} \cdot \eta \in A^{r-1-(k-1)}(M)_{\mathbb{R}}$$

But $L_\ell^{k-1} : A^{k-1}(M)_{\mathbb{R}} \xrightarrow{\sim} A^{r-1-(k-1)}(M)_{\mathbb{R}}$ is an isomorphism.

So there is a <u>unique</u> $\Theta \in A^{k-1}(M)_{\mathbb{R}}$ such that

$$L_\ell^{k-1}(\Theta) = \ell^{r-2k} \cdot \eta.$$

i.e. $$\ell^{r-1-2(k-1)} \cdot \Theta = \ell^{r-2k} \cdot \eta$$

$$\Rightarrow \ell^{r-2k}(\ell \cdot \theta) = \ell^{r-2k} \cdot \eta$$

$$\Rightarrow \ell \cdot L_\ell^k(\ell \cdot \theta) = \ell \cdot L_\ell^k(\eta).$$

Thus, $$\zeta := \eta - \ell \cdot \theta \in \ker(\ell \cdot L_\ell^k) = P_\ell^k$$

and $$\eta = \zeta + \ell \cdot \theta.$$

So $$A^k(M)_{\mathbb{R}} = P_\ell^k \oplus \ell \cdot A^{k-1}(M)_{\mathbb{R}}.$$

Thus, we have the <u>Lefschetz decomposition</u>

$$A^k(M)_{\mathbb{R}} = P_\ell^k \oplus \ell \cdot P_\ell^{k-1} \oplus \ell^2 \cdot P_\ell^{k-2} \oplus \cdots \oplus \ell^k \cdot P_\ell^0.$$

<u>Last time</u>: This decomposition is orthogonal with respect to Q_ℓ^k.

Let's consider 2 special cases:

<u>$k=0$</u>

Here, $A^0(M)_{\mathbb{R}} \cong \mathbb{R}$. (HL) gives the isomorphism

$$L_\ell^0 : A^0(M)_{\mathbb{R}} \longrightarrow A^{r-1}(M)_{\mathbb{R}}$$

which is just mult. by ℓ^{r-1}.

Since $A^r(M)_{\mathbb{R}} = 0$,

$$P_\ell^0(M) = \ker(\underbrace{\ell \cdot L_\ell^0}_{\text{mult. by } \ell^r}) = A^0(M)_{\mathbb{R}}.$$

Now, (HR) tells us that

$$Q_\ell^0 : \underbrace{A^0(M)_{\mathbb{R}} \times A^0(M)_{\mathbb{R}}}_{= \mathbb{R} \times \mathbb{R}} \longrightarrow \mathbb{R}$$

$$(c_1, c_2) \longmapsto \deg(c_1 c_2 \ell^{r-1})$$

$$= c_1 c_2 \deg(\ell^{r-1})$$

is positive-definite on $P_\ell^0 = A^0(M)_{\mathbb{R}}$.

Thus,

$$\boxed{\deg(\ell^{r-1}) > 0}$$

<u>$k = 1$</u>

Here, our Lefschetz decomposition is

$$A^1(M)_{\mathbb{R}} = P_\ell^1 \oplus \ell \cdot \underbrace{P_\ell^0}_{= A^0(M) = \mathbb{R}}$$

$$= P_\ell^1 \oplus \mathbb{R} \cdot \ell .$$

(HR) tells us that Q_ℓ^1 is positive-definite on P_ℓ^1.

On the other hand,

$$\begin{aligned} Q_\ell^1(\ell, \ell) &= (-1)\deg(\ell^{r-3} \cdot \ell \cdot \ell) \\ &= -\deg(\ell^{r-1}) \\ &< 0 \end{aligned}$$

So Q_ℓ^1 is negative-definite on $\mathbb{R}\cdot\ell$.

Now, let $\eta \in A^1(M)_{\mathbb{R}}$ such that $\eta \notin \mathbb{R}\cdot\ell$. Then the restriction of Q_ℓ^1 to

$$\mathrm{span}_{\mathbb{R}}\{\eta, \ell\} \subseteq A^1(M)_{\mathbb{R}}$$

will have matrix

$$\begin{pmatrix} Q_\ell^1(\eta,\eta) & Q_\ell^1(\eta,\ell) \\ Q_\ell^1(\ell,\eta) & Q_\ell^1(\ell,\ell) \end{pmatrix} = \begin{pmatrix} -\deg(\eta^2\ell^{r-3}) & -\deg(\eta\,\ell^{r-2}) \\ -\deg(\eta\,\ell^{r-2}) & -\deg(\ell^{r-1}) \end{pmatrix}.$$

Since the form is indefinite, the determinant must be negative (one positive eigenvalue + one negative eigenvalue).

Thus,

$$\deg(\eta^2 \ell^{r-3}) \cdot \deg(\ell^{r-1}) < \deg(\eta \ell^{r-2})^2$$

Cor: Let $\eta, \ell \in A^1(M)$, with ℓ nef and $\eta \notin \mathbb{R}\cdot\ell$. Then

$$\deg(\eta^2 \ell^{r-3}) \cdot \deg(\ell^{r-1}) \leq \deg(\eta \ell^{r-2})^2.$$

Proof: We already know this when ℓ is ample. If ℓ is nef, then for any ample ℓ' and $\epsilon > 0$,

$$\ell + \epsilon \ell'$$

will be ample.

Apply the above discussion to $\ell + \epsilon \ell'$, then let $\epsilon \to 0$. ∎

Cor: The coefficients μ^k of $\overline{\chi}_M(q)$ are log-concave, i.e.

$$\mu^{k-1}\mu^{k+1} \leq (\mu^k)^2$$

for $1 \leq k \leq r-2$.

__Proof:__ When $k = r-2$, we have

$$\begin{aligned} \mu^{r-3}\mu^{r-1} &= \deg(\alpha^2\beta^{r-3})\deg(\beta^{r-1}) \\ &\leq \deg(\alpha\beta^{r-2})^2 \\ &= (\mu^{r-2})^2 \end{aligned}$$

since β is nef.

Now, if $k \leq r-2$, the coefficient μ^k is unchanged by truncation.

To complete the proof, apply this argument to

$$\mathrm{trunc}(M),\quad \mathrm{trunc}^2(M),\quad \mathrm{trunc}^3(M), \ldots$$

∎

Cor: The unsigned Whitney numbers of the first kind, $|w_i|$, form a log-concave sequence.