The Bergman fan of a matroid

Let M be a loopless matroid on ground set E.

Let $\mathbb{Z}^E$ be the free abelian group with standard basis $\{e_i \mid i \in E\}$.

For each subset $S \subseteq E$, define

$$e_S := \sum_{i \in S} e_i .$$

We let

$$N := \mathbb{Z}^E / \langle e_E \rangle \quad \cong \mathbb{Z}^n / \langle (1,1,\dots,1) \rangle \quad \cancel{\cong \mathbb{Z}^{n-1}}$$

Requires choice of basis

and

$$N_{\mathbb{R}} := N \otimes \mathbb{R} . \quad \cong \mathbb{R}^n / \langle (1,1,\dots,1) \rangle \quad \cancel{\cong \mathbb{R}^{n-1}}$$

__Def__: If $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E)$ is a k-step flag of (non-empty, proper) flats, we let

$$\sigma_{\mathcal{F}} := \left\{ \sum_{i=1}^{k} a_i e_{F_i} \;\middle|\; a_i \in \mathbb{R}_{\geq 0} \right\} \subseteq N_{\mathbb{R}}$$

be the convex cone generated by $e_{F_1}, \dots, e_{F_k}$.

__Note__: $\sigma_{\mathcal{F}}$ is __unimodular__. In particular, $\dim \sigma_{\mathcal{F}} = \#\text{ steps in } \mathcal{F}$.

Def: The <u>Bergman fan</u> of M is

$$\Sigma_M := \{ \sigma_{\mathcal{F}} \mid \mathcal{F} \text{ is a flag of } \underset{\text{non-}\emptyset \text{ proper}}{\wedge} \text{ flats in } M \}.$$

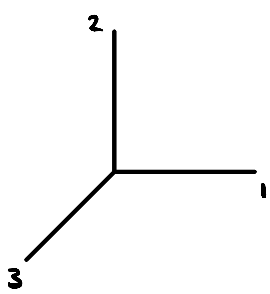
Note: The name "fan" is appropriate:

- Every face of $\sigma_{\mathcal{F}}$ is $\sigma_{\mathcal{F}'}$, where $\mathcal{F}'$ is obtained by removing flats from $\mathcal{F}$.
- $\sigma_{\mathcal{F}} \cap \sigma_{\mathcal{G}}$ is $\sigma_{\mathcal{F}'}$, where $\mathcal{F}'$ is the flag made up of flats in both $\mathcal{F}$ and $\mathcal{G}$.

So Σ_M is a rational polyhedral fan.

Ex: $M = U_{2,3}$.

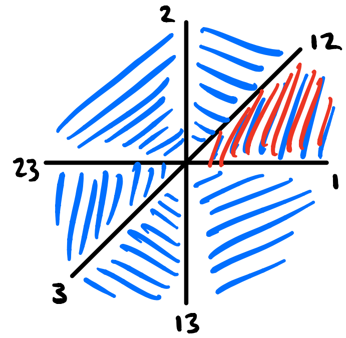
Flags		Cones
$\emptyset \subset 123$	$\longleftrightarrow$	$\{0\}$
$\emptyset \subset i \subset 123$	$\longleftrightarrow$	$\mathbb{R}_{\geq 0} \cdot e_i$

$\Sigma_{U_{2,3}} =$ 

$e_1 + e_2 + e_3 = 0$

Ex: $U_{3,3}$

Flags		Cones
$\emptyset \subset 123$	$\longleftrightarrow$	$\{0\}$
$\emptyset \subset i \subset 123$	$\longleftrightarrow$	$\mathbb{R}_{\geq 0} \cdot e_i$
$\emptyset \subset ij \subset 123$	$\longleftrightarrow$	$\mathbb{R}_{\geq 0} \cdot e_{ij}$
$\emptyset \subset i \subset ij \subset 123$	$\longleftrightarrow$	$\mathbb{R}_{\geq 0} \cdot \langle e_i, e_{ij} \rangle$

$\Sigma_{U_{3,3}} =$ 

Observations

- Maximal cones correspond to complete flags. In particular, each maximal cone has dimension $rk(M)-1$. Therefore, Σ_M is <u>pure</u>.

- The only matroid for which Σ_M is complete is $M = U_{n,n}$.

- Maximal cones in $\Sigma_{U_{n,n}}$ correspond to permutations of $[n]$. Thus, $\Sigma_{U_{n,n}}$ is the normal fan to the $(n-1)$-dimensional <u>permutohedron</u>.

- If M is any matroid on $[n]$, then Σ_M is a subfan of $\Sigma_{U_{n,n}}$

Connectivity in codimension 1

Def: A pure-dimensional fan Σ is <u>connected in codimension 1</u> if for any maximal cones σ and σ', there exists a sequence

$$\sigma \supset \tau_1 \subset \sigma_1 \supset \tau_2 \subset \cdots \supset \tau_{\ell-1} \subset \sigma_{\ell-1} \supset \tau_\ell \subset \sigma'$$

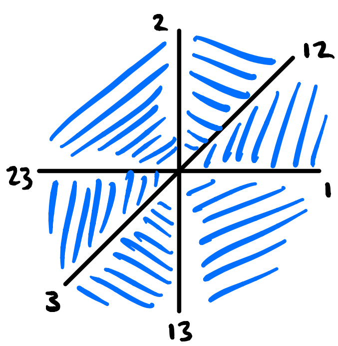
where

- $\sigma_1, \dots, \sigma_{\ell-1}$ are maximal cones
- $\tau_1, \dots, \tau_\ell$ are codimension 1 cones

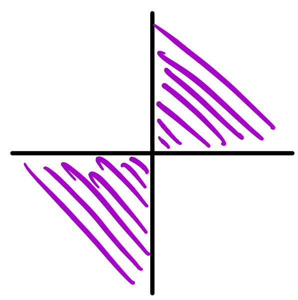
Abbreviate this as $\sigma \sim \sigma'$. Clearly it's an equivalence relation.

Conn. in codim. $\Leftrightarrow$ only one equivalence class.

Ex:



Non-Ex:



Thm: Let M be a loopless matroid of rank r. Then Σ_M is connected in codimension 1.

Proof: We prove this by induction on r.

If $rk(M) = 2$, then Σ_M has dimension 1 and every 1-dimensional fan is connected in codimension 1.

In general, let

$$\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq E)$$

$$\mathcal{G} = (\emptyset \subsetneq G_1 \subsetneq \cdots \subsetneq G_{r-1} \subsetneq E)$$

be maximal flags. We wish to show $\sigma_{\mathcal{F}} \sim \sigma_{\mathcal{G}}$.

Case 1: $F_1 = G_1$.

Apply the inductive hypothesis to M/F_1 to see $\sigma_{\mathcal{F}} \sim \sigma_{\mathcal{G}}$ via a sequence of cones corresponding to flags all starting with F_1.

<u>Case 2</u>: $F_1 \neq G_1$

Then $F_1 \vee G_1 = \mathrm{cl}(F_1 \cup G_1)$ is a rank 2 flat.

Now, let $\mathcal{H}$ be any $(r-2)$-step flag that begins with $F_1 \vee G_1$:

$$\mathcal{H} = \left(\emptyset \subsetneq F_1 \vee G_1 \subsetneq H_3 \subsetneq \cdots \subsetneq H_{r-1} \subsetneq E\right).$$

Then we extend $\mathcal{H}$ to a maximal flag in two ways:

- $\mathcal{H}_{F_1}$ = Add F_1 to $\mathcal{H}$
- $\mathcal{H}_{G_1}$ = Add G_1 to $\mathcal{H}$.

Now, $\sigma_{\mathcal{H}_{F_1}} > \sigma_{\mathcal{H}} < \sigma_{\mathcal{H}_{G_1}}$, so $\sigma_{\mathcal{H}_{F_1}} \sim \sigma_{\mathcal{H}_{G_1}}$.

Thus,

$$\sigma_{\mathcal{F}} \underset{\text{Case 1}}{\sim} \sigma_{\mathcal{H}_{F_1}} \sim \sigma_{\mathcal{H}_{G_1}} \underset{\text{Case 1}}{\sim} \sigma_{\mathcal{G}}.$$

$\blacksquare$

Minkowski weights

If σ is a unimodular cone and $\tau \subset \sigma$ is a facet (codim 1 face), then there is a unique ray (1-dim face) of σ which is not in τ.

Let $e_{\sigma/\tau}$ be the primitive generator of this ray.

Ex: $\sigma = \sigma_{\mathcal{F}}$. A facet τ of σ corresponds to removing a flat F from $\mathcal{F}$. Then

$$e_{\sigma/\tau} = e_F$$

Def: A <u>k-dimensional Minkowski weight</u> on a unimodular fan Σ is a function

$$w : \Sigma_k \longrightarrow \mathbb{Z}$$

k-dimensional cones in Σ

satisfying the <u>balancing condition</u>: for every $(k-1)$-dimensional cone τ in Σ,

$$\sum_{\substack{\sigma \in \Sigma_k \\ \sigma \supset \tau}} w(\sigma)\, e_{\sigma/\tau} \in \mathbb{R} \cdot \tau$$

Important later: The set of all k-dimensional Minkowski weights on Σ, denoted
$$MW_k(\Sigma) \subseteq \mathbb{Z}^{\Sigma_k}$$
is a group under addition.

For now:

Thm: Let M be a loopless matroid of rank r. Then
$$MW_{r-1}(\Sigma_M) \cong \mathbb{Z}.$$
That is, every top-dimensional Minkowski weight is constant.