__Recall:__ A __k-dimensional Minkowski weight__ on a unimodular fan Σ is a function

$$w : \underset{\substack{\uparrow \\ \text{k-dim cones in } \Sigma}}{\Sigma_k} \longrightarrow \mathbb{Z}$$

satisfying the __balancing condition__: for every $(k-1)$-dimensional cone τ in Σ,

$$\sum_{\substack{\sigma \in \Sigma_k \\ \sigma \supset \tau}} w(\sigma)\, e_{\sigma/\tau} \in \mathbb{R} \cdot \tau.$$

We wish to study Minkowski weights on Σ_M, the Bergman fan of a matroid M.

__Recall:__
- For $S \subseteq E$, write $e_S := \sum_{i \in S} e_i \in \mathbb{Z}^E$
- $N := \mathbb{Z}^E / \langle e_E \rangle$, $N_{\mathbb{R}} := N \otimes \mathbb{R}$
- $\Sigma_M \subseteq N_{\mathbb{R}}$ has
  - rays $e_F \leftrightarrow F$ flat (non-$\emptyset$, proper)
  - cones $\sigma_{\mathcal{F}} \leftrightarrow \mathcal{F}$ flag of flats
- Σ_M is unimodular, pure of dim. $r-1$, connected in codimension 1

Thm: Let M be a loopless matroid of rank r. Then

$$MW_{r-1}(\Sigma_M) \cong \mathbb{Z}.$$

That is, every top-dimensional Minkowski weight is constant.

Proof: Let $\tau = \sigma_{\mathcal{F}}$ be a codimension 1 cone.

We need to check the balancing condition around τ.

We know

$$\mathcal{F} = (\underset{\overset{\shortparallel}{F_0}}{\emptyset} \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-2} \subsetneq \underset{\overset{\shortparallel}{F_{r-1}}}{E})$$

is an $(r-2)$-step flag, so there is exactly one index ℓ where

$$rk(F_\ell) = \underbrace{rk(F_{\ell-1})}_{=\ell-1} + 2.$$

Let $G_1, \ldots, G_m$ be the complete list of rank ℓ flats such that

$$F_{\ell-1} \subsetneq G_i \subsetneq F_\ell,$$

and let $\mathcal{F}_i$ be the flag obtained from $\mathcal{F}$ by inserting G_i in position ℓ.

Then $\mathcal{F}_1, \dots, \mathcal{F}_m$ are all of the maximal flags refining $\mathcal{F}$, so $\sigma_{\mathcal{F}_1}, \dots, \sigma_{\mathcal{F}_m}$ are all of the maximal cones containing τ.

If $\omega \in MW_{r-1}(\Sigma_M)$, then the balancing condition says

$$\sum_{i=1}^{m} \omega(\sigma_{\mathcal{F}_i})\, e_{G_i} \in \underbrace{\mathbb{R}\cdot\tau}_{\operatorname{span}\{e_{F_1}, e_{F_2}, \dots, e_{F_{r-2}}\}}$$

Since $F_{\ell-1} \subseteq G_i$ for every i, we can write

$$e_{G_i} = e_{G_i \setminus F_{\ell-1}} + e_{F_{\ell-1}}$$

So the balancing condition becomes

$$\sum_{i=1}^{m} \omega(\sigma_{\mathcal{F}_i})\, e_{G_i \setminus F_{\ell-1}} \in \mathbb{R}\cdot\tau .$$

Since $e_{F_1}, e_{F_2 \setminus F_1}, e_{F_3 \setminus F_2}, \dots, e_{F_{r-2} \setminus F_{r-3}}$ is a basis of $\mathbb{R} \cdot T$, we have

$$(\bigstar) \quad \sum_{i=1}^{m} w(\sigma_{\mathcal{F}_i})\, e_{G_i \setminus F_{\ell-1}} = c_1 e_{F_1} + c_2 e_{F_2 \setminus F_1} + \cdots + c_{r-2} e_{F_{r-2} \setminus F_{r-3}}$$

for some $c_1, \dots, c_{r-2} \in \mathbb{R}$.

Now, by flat axiom (F3) applied to $M|F_\ell$,

$$F_\ell \setminus F_{\ell-1} = \coprod_{i=1}^{m} G_i \setminus F_{\ell-1}.$$

Look again at $(\bigstar)$:

- On the left, this vector has support $F_\ell \setminus F_{\ell-1}$
- On the right, the basis vectors $e_{F_{j+1} \setminus F_j}$ have disjoint support.

So $c_i = 0$ for $i \neq \ell$, and

$$\sum_{i=1}^{m} w(\sigma_{\mathcal{F}_i})\, e_{G_i \setminus F_{\ell-1}} = c_\ell\, e_{F_\ell \setminus F_{\ell-1}}.$$

Since the union

$$F_\ell \setminus F_{\ell-1} = \coprod_{i=1}^{m} G_i \setminus F_{\ell-1}$$

is disjoint, this means $\omega(\sigma_{F_i}) = c_\ell$ for all i.

Now, repeat this argument for all codimension 1 cones.

By connectivity in codimension 1, we must get the same constant weight on every maximal cone. $\blacksquare$

The degree map

We previously asserted that there is an isomorphism

$$\deg : A^{r-1}(M) \longrightarrow \mathbb{Z}$$

such that $\deg(\alpha^{r-1}) = 1$.

Recall: For every complete flag

$$\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq E),$$

we have

$$x_{\mathcal{F}} = x_{F_1} \cdots x_{F_{r-1}} = \alpha^{r-1}.$$

Lemma: For $0 \le k \le r-1$, there is an isomorphism

$$t_k : MW_k(\Sigma_M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A^k(M), \mathbb{Z})$$

$$\omega \longmapsto (x_{\mathcal{F}} \mapsto \omega(\sigma_{\mathcal{F}}))$$

for each k-step flag $\mathcal{F}$

Proof idea: Let

$$w : (\Sigma_M)_k \longrightarrow \mathbb{Z}$$

Check that the balancing condition is precisely the statement that

$$x_F \longmapsto w(\sigma_F)$$

is well-defined modulo the relations on $A^\bullet(M)$. $\blacksquare$

This lets us define a cap product

$$A^k(M) \times MW_\ell(\Sigma_M) \longrightarrow MW_{\ell-k}(\Sigma_M)$$

$$(\eta, w) \longmapsto \eta \frown w := \Big(\sigma_F \mapsto \underbrace{(t_k w)}_{\in \mathrm{Hom}(A^\ell, \mathbb{Z})}\underbrace{(\eta \cdot x_F)}_{\in A^\ell(M)}\Big)$$

We've shown $MW_{r-1}(\Sigma_M) \cong \mathbb{Z}$.

Clearly, $MW_0(\Sigma_M) \cong \mathbb{Z}$ also.

Def: The degree map is

$$\deg : A^{r-1}(M) \longrightarrow MW_0(\Sigma_M) \cong \mathbb{Z}.$$

$$\eta \longmapsto \eta \frown 1$$

We just need to check that $\alpha^{r-1} \frown 1 = 1$.

Let $\mathcal{F}$ be any complete flag, so

$$\alpha^{r-1} = x_{\mathcal{F}}.$$

Then $\alpha^{r-1} \frown 1 = x_{\mathcal{F}} \frown 1$ is the 0-dim Minkowski weight

$$\underbrace{(t_{r-1} 1)}_{\mathrm{Hom}(A^{r-1}, \mathbb{Z})} (\underbrace{x_{\mathcal{F}}}_{A^{r-1}}) = 1(\sigma_{\mathcal{F}}) = 1. \qquad \checkmark$$

Why should we care about Σ_M?

Perspective 1: Tropical geometry

Let Σ be a pure d-dimensional rational polyhedral fan, and $\omega: \Sigma_d \to \mathbb{Z}_{>0}$ a positive weighting of its top-dimensional cones.

Then (Σ, ω) is a tropical variety if

- Σ is connected in codimension 1
- $\omega \in MW_d(\Sigma)$, i.e. (Σ, ω) is "balanced"

So we've shown that $(\Sigma_M, 1)$ is a tropical variety.

- If $M = M(A)$ is representable (over any field), then
$$\mathrm{Trop}(\mathbb{P}U_A) = (\Sigma_M, 1)$$
- [Katz-Payne '11] Conversely, $\mathrm{Trop}(X) = (\Sigma_M, 1)$ for some K-variety X if and only if M is K-representable.
- [Fink '13] In a precise sense, Bergman fans are precisely the "linear" tropical varieties.

Perspective 2: Toric Geometry

The fan Σ_M defines a toric variety $X(\Sigma_M)$.

- Σ_M is unimodular $\Rightarrow$ $X(\Sigma_M)$ is smooth
- If $M \neq U_{n,n}$, then Σ_M is not complete $\Rightarrow$ $X(\Sigma_M)$ is not compact

If M has ground set $[n]$, then Σ_M is a subfan of $\Sigma_{U_{n,n}}$. This gives an open embedding

$$X(\Sigma_M) \hookrightarrow X(\Sigma_{U_{n,n}}).$$

The variety $X(\Sigma_{U_{n,n}})$ is the <u>permutohedral variety</u>.

- Consider the arrangement of all n coordinate hyperplanes in $\mathbb{P}^{n-1}$.

  The wonderful compactification of the complement $(=\mathbb{G}_m^{n-1})$ is $X(\Sigma_{U_{n,n}})$.

- Suppose M is K-representable. Let $\mathcal{A}$ be a configuration in a vector space V such that $M = M(\mathcal{A})$.

  Then we have $K^n \twoheadrightarrow V$, or dually
$$V^* \hookrightarrow K^n.$$
  Thus,
$$\mathbb{P}V^* \hookrightarrow \mathbb{P}^{n-1}.$$

  If $\pi : X(\Sigma_{U_{n,n}}) \longrightarrow \mathbb{P}^{n-1}$ is the natural projection, then $\pi^{-1}(\mathbb{P}V^*)$ is the wonderful compactification $Y_{\mathcal{A}}$:

$$\begin{tikzcd} Y_{\mathcal{A}} \arrow[r, hook] \arrow[d] & X(\Sigma_{U_{n,n}}) \arrow[d, "\pi"] \\ \mathbb{P}V^* \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

- The <u>Cremona transform</u> is the rational map

$$\mathrm{Crem}: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$$
$$(x_0 : \cdots : x_n) \longmapsto \left(\frac{1}{x_0} : \cdots : \frac{1}{x_n}\right)$$

This extends to an automorphism of the permutohedral variety:

$$\begin{tikzcd} X(\Sigma_{U_{n,n}}) \arrow[r, "\widetilde{\mathrm{Crem}}"] \arrow[d, "\pi"] & X(U_{n,n}) \arrow[d, "\pi"] \\ \mathbb{P}^n \arrow[r, dashed, "\mathrm{Crem}"] & \mathbb{P}^n \end{tikzcd}$$

If M is representable, so the wonderful compactification $Y_{\mathcal{A}}$ embeds

$$\iota_{\mathcal{A}} : Y_{\mathcal{A}} \hookrightarrow X(\Sigma_{U_{n,n}})$$

Then

- the pullback of the hyperplane class in $\mathbb{P}^n$ under $\pi \circ \iota_{\mathcal{A}}$ is $\alpha \in A^1(M)$.
- the pullback of the hyperplane class under $\pi \circ \widetilde{\mathrm{Crem}} \circ \iota_{\mathcal{A}}$ is $\beta \in A^1(M)$.

- [Feichtner-Yuzvinsky '04] In the above situation, the image of Y_A is contained in $X(\Sigma_M)$. Moreover,

$$\iota_A : Y_A \hookrightarrow X(\Sigma_M)$$

is a <u>Chow equivalence</u>: the induced map

$$\iota_A^* : A^\bullet(X_M) \longrightarrow A^\bullet(Y_A) = A^\bullet(M)$$

is an isomorphism.

- Actually, $A^\bullet(X_M) \cong A^\bullet(M)$ for <u>all</u> loopless matroids.

  Curiously, the Kähler package tells us that $A^\bullet(M)$ behaves like ~~the~~ the Chow ring of a smooth, projective variety of dimension $r-1$.

  $X(\Sigma_M)$, by contrast, is smooth but not projective, and has dimension $n-1$.

- [Adiprasito - Huh - Katz '15] If Y is a smooth projective K-variety and

$$Y \longrightarrow X(\Sigma_M)$$

is a Chow equivalence, then M is K-representable.