

Abbott: 1.2.5, 1.2.9, 1.2.12, 1.3.3

**1.2.5 (De Morgan's Laws)** Let  $A$  and  $B$  be subsets of  $\mathbb{R}$ .

- (a) If  $x \in (A \cap B)^c$ , explain why  $x \in A^c \cup B^c$ . This shows that  $(A \cap B)^c \subseteq A^c \cup B^c$ .
- (b) Prove the reverse inclusion  $(A \cap B)^c \supseteq A^c \cup B^c$ , and conclude that  $(A \cap B)^c = A^c \cup B^c$ .
- (c) Show  $(A \cup B)^c = A^c \cap B^c$  by demonstrating inclusion both ways.

**1.2.9** Given a function  $f: D \rightarrow \mathbb{R}$  and a subset  $B \subseteq \mathbb{R}$ , let  $f^{-1}(B)$  be the set of all points from the domain  $D$  that get mapped into  $B$ ; that is,  $f^{-1}(B) = \{x \in D : f(x) \in B\}$ . This set is called the *preimage* of  $B$ .

- (a) Let  $f(x) = x^2$ . If  $A$  is the closed interval  $[0, 4]$  and  $B$  is the closed interval  $[-1, 1]$ , find  $f^{-1}(A)$  and  $f^{-1}(B)$ . Does  $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$  in this case? Does  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ ?
- (b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function  $g: \mathbb{R} \rightarrow \mathbb{R}$ , it is always true that  $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$  and  $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$  for all sets  $A, B \subseteq \mathbb{R}$ .

**1.2.12** Let  $y_1 = 6$ , and for each  $n \in \mathbb{N}$  define  $y_{n+1} = (2y_n - 6)/3$ .

- (a) Use induction to prove that the sequence satisfies  $y_n > -6$  for all  $n \in \mathbb{N}$ .
- (b) Use another induction argument to show the sequence  $(y_1, y_2, y_3, \dots)$  is decreasing.

**1.3.3**

(a) Let  $A$  be nonempty and bounded below, and define

$$B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}.$$

Show that  $\sup B = \inf A$ .

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.