Math 315
Homework #14
5/30/2017

Abbott: 4.5.3, 4.5.7, 5.2.7, 5.2.10

4.5.3 A function f is *increasing* on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

4.5.7 Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

5.2.7 Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0, 1]$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

5.2.10 Recall that a function $f\colon (a,b) \to \mathbb{R}$ is *increasing* on (a,b) if $f(x) \leq f(y)$ whenever $x < y$ in (a,b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.

Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Now, prove that g is *not* increasing over any open interval containing 0.

In the next section we will see that f is indeed increasing on (a,b) if and only if $f'(x) \geq 0$ for all $x \in (a,b)$.