Math 315
Homework #16
6/6/2017

Abbott: 6.2.1, 6.2.3 (Extra Credit: 6.2.12)

6.2.1 Let
$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.

(b) Is the convergence uniform on $(0, \infty)$?

(c) Is the convergence uniform on $(0, 1)$?

(d) Is the convergence uniform on $(1, \infty)$?

6.2.3 For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1 + x^n} \qquad \text{and} \qquad h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n. \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n):

(a) Find the pointwise limit on $[0, \infty)$.

(b) Explain how we know that the convergence *cannot* be uniform on $[0, \infty)$.

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

6.2.12 (Extra Credit) Review the construction of the Cantor set $C \subseteq [0,1]$ from Section 3.1. This exercise makes use of results and notation from this discussion.

(a) Define $f_0(x) = x$ for all $x \in [0,1]$. Now, let

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \leq x \leq 1/3 \\ 1/2 & \text{for } 1/3 < x < 2/3 \\ (3/2)x - 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

Sketch f_0 and f_1 over $[0,1]$ and observe that f_1 is continuous, increasing, and is constant on the middle third $(1/3, 2/3) = [0,1] \setminus C_1$.

(b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, let

$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_1(3x-2) + 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

If we continue this process, show that the resulting sequence (f_n) converges uniformly on $[0,1]$.

(c) Let $f = \lim f_n$. Prove that f is a continuous, increasing function on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0,1] \setminus C$. Recall that the "length" of the Cantor set C is 0. Somehow, f manages to increase from 0 to 1 while remaining constant on a set of "length 1."