MATROID THEORY: HOMEWORK 1

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This assignment is due on Monday, February 5.

1. **The matroid of a hyperplane arrangement**
   Let V be a finite-dimensional vector space over a fixed field K. A **linear hyperplane** in V is a linear subspace of V of codimension 1. A **hyperplane arrangement** $\mathcal{A}$ in V is a collection

$$\mathcal{A} = \{H_e \mid e \in E\},$$

   where E is a finite indexing set, and each H_e is either a linear hyperplane in V or the **degenerate hyperplane** $H_e = V$. We allow repeated hyperplanes ($H_e = H_f$ for $e \neq f$). Let $\mathcal{I} \subseteq 2^E$ be the collection of subsets $I \subseteq E$ such that

$$\operatorname{codim}\left(\bigcap_{e \in I} H_e\right) = |I|.$$

   (a) Prove that $(E, \mathcal{I})$ is a matroid, which we shall denote $M(\mathcal{A})$. (This is the same notation for the matroid of an arrangement of *vectors*. It will generally be clear from the context whether $\mathcal{A}$ is a hyperplane arrangement or a vector arrangement.)
   (b) Identify the bases, rank function, and flats of $M(\mathcal{A})$.
   (c) Identify the loops and spanning sets of $M(\mathcal{A})$.
   (d) Prove that a matroid is representable (as the matroid of a vector arrangement) if and only if it is isomorphic to the matroid of a hyperplane arrangement.

2. **Representability of uniform matroids**
   (a) Prove that $U_{2,4}$ is representable over a field K if and only if $|K| \geq 3$.
   (b) Fix $n \geq 2$. Find necessary and sufficient conditions on K for $U_{2,n}$ to be K-representable.
   (c) Prove that $U_{2,4}$ is not graphic.
   (d) Find all pairs (r, n), $0 \leq r \leq n$, such that $U_{r,n}$ is graphic.

3. **Circuit-hyperplane relaxation**
   Let M be a matroid on E, and suppose $H \subseteq E$ is a hyperplane which is also a circuit.
   (a) Prove that $\mathcal{B}' = \mathcal{B}(M) \cup \{H\}$ is the set of bases of a matroid M' on E.
   (b) Identify the circuits of M'.

4. **Loops**
   Let M be a matroid on E, and let $e \in E$. Prove the equivalence of the following statements.
   (a) $\{e\}$ is a circuit
   (b) e is in no bases.
   (c) e is in no independent sets.
   (d) If $X \subseteq E$ is any subset, then $e \in \mathrm{cl}(X)$.
   (e) $e \in \mathrm{cl}(\emptyset)$.
   (f) $\mathrm{rk}(\{e\}) = 0$.
   We say that e is a **loop** of M if it satisfies any of the equivalent conditions (a)–(f).

5. **Characterization of matroids by flats**
   Let E be a finite set and $\mathcal{F} \subseteq 2^E$ a collection of subsets. Prove that $\mathcal{F}$ is the set of flats of a matroid on E if and only if $\mathcal{F}$ satisfies the following conditions:
   (F1) $E \in \mathcal{F}$.
   (F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.
   (F3) If $F \in \mathcal{F}$ and $\{F_1, \ldots, F_k\}$ is the set of minimal members of $\mathcal{F}$ properly containing F, then $\{F_1 \setminus F,\, F_2 \setminus F,\, \ldots,\, F_k \setminus F\}$ is a partition of $E \setminus F$.