MATROID THEORY: HOMEWORK 2

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This assignment is due on Monday, February 26.

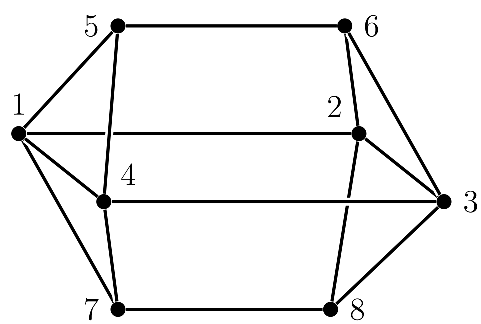
1. **Paving matroids**
   Note that every circuit in a matroid M has size at most $\mathrm{rk}(M) + 1$ (Why?). We say M is **paving** if it has no circuits of size less than $\mathrm{rk}(M)$.
   (a) Show that a matroid M is uniform if and only if it has no circuits of size less than $\mathrm{rk}(M) + 1$. Conclude that uniform matroids are paving.
   (b) Let $\mathcal{D}$ be a collection of subsets of a finite set E with $\emptyset \notin \mathcal{D}$. Prove that $\mathcal{D}$ is the set of circuits of a paving matroid on E if and only if there exists a subset $\mathcal{D}' \subseteq \mathcal{D}$ such that
   (P1) There is an integer $k \leq |E|$ such that every $D \in \mathcal{D}'$ has $|D| = k$;
   (P2) If $D_1 \neq D_2$ are distinct members of $\mathcal{D}'$ with $|D_1 \cap D_2| = k - 1$, then each k-element subset of $D_1 \cup D_2$ is in $\mathcal{D}'$;
   (P3) $\mathcal{D} \setminus \mathcal{D}' = \{X \subseteq E \mid |X| = k + 1 \text{ and no subset of } X \text{ is in } \mathcal{D}'\}$.
   (c) Let d be a positive integer. A collection $\mathcal{T}$ of at least two subsets of E is called a d**-partition** if
      (i) Every $T \in \mathcal{T}$ has $|T| \geq d$; and
      (ii) If $X \subseteq E$ has size d, then there is a unique $T \in \mathcal{T}$ such that $X \subseteq T$.
   Prove that $\mathcal{T} \subseteq 2^E$ is a d-partition if and only if $\mathcal{T}$ is the set of hyperplanes of a paving matroid on E of rank $d + 1$.

2. **The poset dual of a lattice**
   Let $\mathcal{P}$ be a poset. Let $\mathcal{P}'$ be the poset defined by $x \leq y$ in $\mathcal{P}'$ if and only if $x \geq y$ in $\mathcal{P}$. In other words, the Hasse diagram for $\mathcal{P}'$ is obtained by flipping the Hasse diagram for $\mathcal{P}$ upside-down.
   (a) Prove that if $\mathcal{P}$ is a lattice, then so is $\mathcal{P}'$.
   (b) Show, by examples, that if $\mathcal{P}$ is a geometric lattice, then $\mathcal{P}'$ may or may not be geometric.

3. **Circuit-cocircuit orthogonality**
   Let M be a matroid on E. Prove that $X \subseteq E$ is a circuit of M if and only if X is a minimal non-empty set with the property that $|X \cap C^*| \neq 1$ for every cocircuit C^* of M.

4. **Fundamental cocircuits**
Let M be a matroid on E. If B is a basis of M and $e \in B$, then we have the fundamental circuit $C_{M^*}(e, E \smallsetminus B)$ in M^*. Considered as a cocircuit of M, this is called the **fundamental cocircuit** of e with respect to B and is denoted $C^*(e, B)$.
   (a) Show that $C^*(e, B)$ is the unique cocircuit of M that is disjoint from $B \smallsetminus e$.
   (b) For $f \in E \smallsetminus B$, show that $f \in C^*(e, B)$ if and only if $e \in C(f, B)$.
   (c) Let $C_1^*, \ldots, C_r^*$ be distinct cocircuits of a rank-r matroid M. Prove that the following statements are equivalent.
      (i) For each $j = 1, \ldots, r$, the cocircuit C_j^* is not contained in $\bigcup_{i \neq j} C_i^*$.
      (ii) There is a basis B of M such that $C_1^*, \ldots, C_r^*$ is a complete list of fundamental circuits with respect to B.

5. **The Vámos matroid**
The Vámos matroid V_8 is the matroid on 8 elements with the following geometric representation.



Note that, as is typical in such representations, we have only drawn the "interesting" coplanarities: the 5 sets of four coplanar points. Importantly, the points $\{5, 6, 7, 8\}$ are *not* coplanar.
   (a) Use exercise 1(c) to prove that V_8 is a rank 4 paving matroid.
   (b) Prove that V_8 is self-dual but not identically self-dual.
   (c) Prove that V_8 is not representable over any field.
   To get you started: The goal is to show that in any such configuration of points, $\{5, 6, 7, 8\}$ must be coplanar. Do this by showing that the lines $\overline{56}$ and $\overline{78}$ must intersect. As a first step, show that $\overline{56}$ must intersect the plane containing the points $\{1, 2, 3, 4\}$.