MATROID THEORY: HOMEWORK 3

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This assignment is due on Wednesday, March 28.

1. **Separators**
   Let M be a matroid on E. Recall that a **separator** of M is a subset $T \subseteq E$ satisfying one of the following equivalent properties.
   (a) $M \backslash T = M/T$
   (b) $\mathrm{rk}(M \backslash T) = \mathrm{rk}(M/T)$
   (c) $\mathrm{rk}(T) + \mathrm{rk}(E \backslash T) = \mathrm{rk}(M)$
   (d) $\mathrm{rk}(T) + \mathrm{rk}^*(T) = |T|$
   (e) If $C \in \mathcal{C}(M)$ is a circuit, then $C \subseteq T$ or $C \subseteq E \smallsetminus T$.
   (f) T is a union of connected components of M
   (g) $M = M|_T \oplus M|_{E \smallsetminus T}$
   Prove the equivalence of (a)–(g).

2. **Graphic matroids are regular**
   Let G be a graph.
   (a) Produce a matrix A_G such that $M(G) \cong M(A_G)$, where $M(A_G)$ is the matroid given by the arrangement of column vectors in A_G. Conclude that $M(G)$ is representable over every field.
   (b) Prove A_G is totally unimodular when it is viewed as a matrix over $\mathbb{R}$.

3. **Regular matroids are binary**
   Let A be a totally unimodular matrix. Since every entry of A is in $\{-1, 0, 1\}$, we may reduce each entry modulo 2 to obtain a matrix $A^{\#}$ over $\mathbb{F}_2$. Prove $M(A) \cong M(A^{\#})$.

4. **Binary matroids**
   Let M be a matroid. Prove that the following statements are equivalent *without using the excluded-minor characterization of binary matroids.*
   (a) M is binary.
   (b) For every circuit $C \in \mathcal{C}(M)$ and cocircuit $C^* \in \mathcal{C}^*(M)$, $|C \cap C^*|$ is even.
   (c) For every circuit $C \in \mathcal{C}(M)$ and cocircuit $C^* \in \mathcal{C}^*(M)$, $|C \cap C^*| \neq 3$.
   (d) For every basis $B \in \mathcal{B}(M)$ and circuit $C \in \mathcal{C}(M)$, $C = \Delta_{e \in C \smallsetminus B} C(e, B)$, where Δ denotes symmetric difference of sets.

5. **The missing step in our excluded-minor characterization of binary matroids** Let M_1 and M_2 be matroids on the same ground set E. Suppose that $B \subseteq E$ is a basis for both M_1 and M_2, and furthermore that the fundamental circuits $C_{M_1}(e, B)$ and $C_{M_2}(e, B)$ agree for every $e \in E \smallsetminus B$.
   (a) Show that M_1 and M_2 are not necessarily equal.
   (b) Show that if M_1 and M_2 are both binary, then $M_1 = M_2$.