MATROID THEORY: HOMEWORK 4

MAX KUTLER

This assignment is due on Wednesday, April 18.

1. **Bases of maximal weight**
   Let M be a matroid on $[n]$, and let $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$ be a weight vector. The w**-weight** of a basis $B \in \mathcal{B}(M)$ is the sum $\sum_{i \in B} w_i$. Let $\mathcal{B}_w \subseteq \mathcal{B}(M)$ be the collection of all bases of M of maximal w-weight.
   (a) Prove that the greedy algorithm always outputs a member of $\mathcal{B}_w$.
   (b) Prove that $\mathcal{B}_w$ is the collection of bases of a matroid M_w on $[n]$.

2. **The Bergman fan of a matroid**
   Note that a weight vector $w \in \mathbb{R}^n$ uniquely defines a flag of subsets of $[n]$
   $$\mathscr{F}(w) = \big(\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = [n]\big)$$
   satisfying the two conditions
   - w is constant on each difference $F_{i+1} \smallsetminus F_i$
   - w is strictly decreasing: $w|_{F_i \smallsetminus F_{i-1}} > w|_{F_{i+1} \smallsetminus F_i}$.

   For example, if $n = 6$ and $w = (1, 7, 1, 1, 2, 7)$, then the flag $\mathscr{F}(w)$ is
   $$\emptyset \subsetneq \{2, 6\} \subsetneq \{2, 5, 6\} \subsetneq [6].$$
   Prove that the following are equivalent for a loopless matroid M on $[n]$ and a weight vector $w \in \mathbb{R}^n$
   (a) For every circuit $C \in \mathcal{C}(M)$, the minimum value in the set $\{w_i \mid i \in C\}$ is achieved at least twice.
   (b) The matroid M_w is loopless.
   (c) Each set F_i in the flag $\mathscr{F}(w)$ is a flat of M.

3. **The fine subdivision is finer than the coarse subdivision**
   Let M be a (loopless) matroid on $[n]$ and $w \in \mathbb{R}^n$.
   (a) Prove that the flag $\mathscr{F}(w)$ completely determines the matroid M_w. Thus, if $\mathscr{F}(w) = \mathscr{F}(w')$, then $M_w = M_{w'}$.
   (b) Provide an example of a matroid M and weight vectors w and w' such that $M_w = M_{w'}$, but $\mathscr{F}(w) \neq \mathscr{F}(w')$.

4. **The independence complex of $\widetilde{B}(M)$**
   Let M be a loopless matroid on $[n]$, and $S \subseteq [n]$.
   (a) Suppose $S = \{s_1, \ldots, s_k\}$ is independent in M. Produce a flag $\mathscr{F}$ of flats of M such that the projection of the cone $C_{\mathscr{F}}$ onto $\mathbb{R}^S$ has full dimension $|S| = k$.
   (b) Conversely, suppose S is dependent. Show that, for any flag $\mathscr{F}$ of flats of M, the projection of $C_{\mathscr{F}}$ onto $\mathbb{R}^S$ has dimension strictly less than $|S|$.