MATROID THEORY: HOMEWORK 5

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This assignment is due on Wednesday, May 2.

1. **The coefficients of the Tutte polynomial**
   Let M be a matroid on E. Choose, arbitrarily, a total order $\prec$ on E (e.g. enumerate $E = \{e_1, \ldots, e_n\}$ and set $e_i \prec e_j$ if and only if $i < j$). With respect to a given basis B of M we say that $e \in E$ is
   - **internally active** if $e \in B$ and e is the smallest element in $C^*(e, B)$;
   - **externally active** if $e \notin B$ and e is the smallest element in $C(e, B)$.

   Then we define the **internal activity** of B to be the number of $e \in B$ which are internally active, and similarly the **external activity** of B to be the number of $e \notin B$ which are externally active.

   Prove that
   $$T_M(x, y) = \sum_{i,j} t_{i,j} x^i y^j,$$
   where $t_{i,j}$ is the number of bases of M which have internal activity i and external activity j. Conclude that the numbers $t_{i,j}$ are independent of the total order $\prec$ on E.

2. **The coefficients of T_M (continued)**
   Let M be a matroid on E Let $t_{i,j}$ be the coefficient of $x^i y^j$ in T_M, as above. Prove the following.
   (a) $t_{0,0} = 0$ if and only if $M = U_{0,0}$.
   (b) $t_{1,0} \neq 0$ if and only if M is loopless and connected. The number $t_{1,0}$ is called the **beta invariant** of M, denoted $\beta(M)$.
   (c) If $|E| > 1$, then $t_{1,0} = t_{0,1}$. Conclude that $\beta(M) = \beta(M^*)$ for all matroids on at least two elements.
   (d) Suppose M has exactly ℓ loops and k coloops. Then $t_{i,j} = 0$ if $i \geq \mathrm{rk}(M)$ or $j \geq \mathrm{null}(M)$, with the exception that $t_{\mathrm{rk}(M),\ell} = 1 = t_{k,\mathrm{null}(M)}$.

3. **The convolution formula for T_M**
   Let M be a matroid on E. Show that
   $$T_M(x, y) = \sum_{S \subseteq E} T_{M/S}(x, 0) T_{M|_S}(0, y)$$
   where M/S and $M|_S = M \smallsetminus (E \smallsetminus S)$ denote, respectively, the contraction of and restriction to S.

4. **Acylic orientations** Let M be a graphic matroid. For simplicity, let G be a connected graph with $M \cong M(G)$.

   An orientation of G is **acyclic** if the resulting directed graph has no directed circuits. On the other hand, an orientation of G is totally cyclic if every edge is contained in some directed circuit.

   In an orientation of G, a vertex v is a **source** if that every edge incident to v is outgoing, and a **sink** if every edge incident to v is incoming.

   (a) Prove that the number of acyclic orientations of G is a Tutte-Grothendieck invariant, and that it is equal to $T_M(2,0)$.
   (b) Prove that the number of totally cyclic orientations of G is $T_M(0,2)$.
   (c) Prove that every acyclic orientation has at least one source and at least one sink.
   (d) Let v be any vertex in G. Prove that the number of acyclic orientations of G with v as the unique source is a Tutte-Grothendieck invariant, and express it as an evaluation of the Tutte polynomial. Conclude that this number is independent of the choice of v.
   (e) Let $e \in E(G)$ be an edge with endpoints v and u. Prove that the number of acyclic orientations of G with unique source v and unique sink u is the beta invariant $\beta(M(G))$.
   (f) BONUS PROBLEM: Formulate and prove statements analogous to (c)–(e) for totally cyclic orientations. In particular, find an interpretation of the beta invariant in terms of totally cyclic orientations.