

HOMEWORK 14
MATH 3345 – SPRING 2024 – KUTLER

Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. Let $a, b \in \mathbb{N}$. We say that a positive integer $m \in \mathbb{N}$ is a **common multiple** of a and b if $a|m$ and $b|m$.
 - (a) Show that for any $a, b \in \mathbb{N}$, ab is a common multiple of a and b .
 - (b) Prove that for any $a, b \in \mathbb{N}$, there exists a common multiple ℓ of a and b such that $\ell \leq m$ if m is any common multiple of a and b . This number ℓ is called the **least common multiple** of a and b . We write $\ell = \text{lcm}(a, b)$.
 - (c) Give an example of positive integers $a, b \in \mathbb{N}$ such that $\text{lcm}(a, b) = ab$.
 - (d) Give an example of positive integers $a, b \in \mathbb{N}$ such that $\text{lcm}(a, b) < ab$.
 - (e) Explain why there do not exist positive integers a and b such that $\text{lcm}(a, b) > ab$.
2. [**Falkner Section 4 Exercise 25**] Let $m \in \mathbb{N}$. Show that
 - (a) For all $a \in \mathbb{Z}$, we have $a \equiv a \pmod{m}$. [**Reflexivity**]
 - (b) For all $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$. [**Symmetry**]
 - (c) For all $a, b, c \in \mathbb{Z}$, if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$. [**Transitivity**]
3. [**Falkner Section 4 Exercise 26 – modified**] Let $m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$. Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
 - (a) Prove that $a + c \equiv b + d \pmod{m}$.
 - (b) Prove that $a - c \equiv b - d \pmod{m}$.
 - (c) Prove that $ac \equiv bd \pmod{m}$. [HINT: Since $a \equiv b \pmod{m}$, m divides $b - a$, so $b - a = mk$ for some integer k . Rewrite this as $b = a + mk$. Similarly, $d = c + m\ell$ for some integer ℓ .]

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. Let $a, b \in \mathbb{N}$. Prove that $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$.
2. **[Falkner Section 4 Exercise 16]** Let $n \in \mathbb{N}$. Prove that there exists a prime number q such that $n < q \leq 1 + n!$. [HINT: Take q to be any prime which divides $1 + n!$. (How do we know such a prime exists?) Now explain why $q \leq 1 + n!$ and $q > n$ must both be true.]