Homework 15

Math 3345 – Spring 2024 – Kutler

Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. Without using a calculator, find the natural number k such that $0 \leq k \leq 14$ and k satisfies the given congruence.
   (a) $2^{75} \equiv k \mod 15$
   (b) $6^{41} \equiv k \mod 15$
   (c) $140^{874} \equiv k \mod 15$
2. Let $a, b, c \in \mathbb{N}$. Prove that if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$.

   [HINT: First check that the statement is true if any of a, b, or c is equal to 1. Then, for the case where $a > 1$, $b > 1$, and $c > 1$, consider unique prime factorizations.]

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. Recall that any positive integer $n \in \mathbb{N}$ has a unique **base-10 expression:**

   $$n = \sum_{i=0}^{k} a_i \, 10^i,$$

   where $k \geq 0$ and $0 \leq a_i \leq 9$ for all i. The integers a_i are the **digits** of n. For example,

   $$4592 = 2 \cdot 10^0 + 9 \cdot 10^1 + 5 \cdot 10^2 + 4 \cdot 10^3.$$

   Prove the following:

   (a) $2|n$ if and only if 2 divides the "ones digit" a_0.
   (b) $3|n$ if and only if 3 divides the sum of the digits $\sum_{i=0}^{k} a_i$.
   (c) $5|n$ if and only if the ones digit a_0 is equal to 0 or 5.

2. Formulate and prove divisibility criteria similar to those in the previous problem for the following conditions:

   (a) $4|n$.

   (b) $9|n$.

   (c) $11|n$.