Homework 20

Math 3345 – Spring 2024 – Kutler

Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. **[Falkner Section 10 Exercise 19 – modified]** Let A, B, and X be sets.

   (a) Prove that if $A \subseteq B$, then $X \setminus B \subseteq X \setminus A$.

   (b) Prove that $A \subseteq X$ if and only if $A = X \setminus (X \setminus A)$. [HINT: Use Homework 19 Exercise 4 to express $X \setminus (X \setminus A)$ in a simpler form.]

   (c) Suppose $A \subseteq X$. Prove that if $X \setminus B \subseteq X \setminus A$, then $A \subseteq B$.

   (d) Show, by giving an example, that the implication

   $$\text{if } X \setminus B \subseteq X \setminus A, \text{ then } A \subseteq B$$

   may be **false** if $A \not\subseteq X$.

   That is, give an example of sets A, B, and X such that $X \setminus B \subseteq X \setminus A$ and $A \not\subseteq B$.

2. **[Falkner Section 10 Exercise 33(a)–(d)]** Let A, B, C, and D be sets.

   (a) Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

   (b) Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$ and $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

   (c) Prove that $(C \times D) \setminus (A \times B) = E \cup F$, where $E = (C \setminus A) \times D$ and $F = C \times (D \setminus B)$.

   (d) In the special case where $A = [1, 3] = B$ and $C = [2, 4] = D$, draw a picture to illustrate the result you proved for the general case in part (c).

3. **[Falkner Section 11 Exercise 6]** Consider the following functions.

   $$\begin{aligned} f &\colon \mathbb{R} \to \mathbb{R} \\ x &\mapsto x^2 + 1 \end{aligned} \qquad \begin{aligned} g &\colon [1, \infty) \to \mathbb{R} \\ y &\mapsto \sqrt{y - 1} \end{aligned} \qquad \begin{aligned} h &\colon [2, 3) \to \mathbb{R} \\ u &\mapsto 1 - u \end{aligned}$$

   Find the range of f, the range of g, and the range of h.

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. **[Falkner Section 10 Exercise 24]** Prove Proposition 10.34(b): Let $\mathscr{A}$ be a nonempty set of sets and let X be any object. Then

$$x \notin \left(\bigcap_{A \in \mathscr{A}} A \right) \text{ if and only if there exists } A \in \mathscr{A} \text{ such that } x \notin A.$$

2. **[Falkner Section 10 Exercise 32]** Sketch the rectangle $[1, 4] \times [2, 3]$ in the coordinate plane. (Shade the set of points that belong to this rectangle.