Homework 22
Math 3345 – Spring 2024 – Kutler

Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. [**Falkner Section 11 Exercise 15(b) – modified**] Let S and T be sets. Prove that S and T are disjoint (i.e., $S \cap T = \varnothing$) if and only if the following condition holds:

   For all subsets $A_1, A_2 \subseteq S$ and $B_1, B_2 \subseteq T$, if $A_1 \cup B_1 = A_2 \cup B_2$, then $A_1 = A_2$ and $B_1 = B_2$. $(\star)$

   Note: The condition $(\star)$ is equivalent to the statement that the function

   $$\begin{aligned} f \colon \mathscr{P}(S) \times \mathscr{P}(T) &\to \mathscr{P}(S \cup T) \\ (A, B) &\mapsto A \cup B \end{aligned}$$

   is injective. You proved on Homework 21 that this function is always surjective.

2. [**Falkner Section 11 Exercise 26**] Let A, B, and C be sets. Prove that if $f \colon A \to B$ and $g \colon B \to C$ are bijections, then $g \circ f \colon A \to C$ is a bijection.

3. [**Falkner Section 11 Exercise 17 – modified**] Let

   $$\begin{aligned} f \colon [1, \infty) &\to \mathbb{R} \\ x &\mapsto x - 1. \end{aligned}$$

   (a) Show that $\operatorname{Rng}(f) \subseteq [0, \infty)$. That is, $f(x) \in [0, \infty)$ for every $x \in [1, \infty)$.

   (b) Prove that $\operatorname{Rng}(f) = [0, \infty)$. [HINT: In light of part (a), you need only prove the other inclusion, $[0, \infty) \subseteq \operatorname{Rng}(f)$. That is, for each $y \in [0, \infty)$, you must find some $x \in \operatorname{Dom}(f) = [1, \infty)$ such that $f(x) = y$.]

   (c) Prove that f is an injection.

   (d) Conclude that f is a bijection from $[1, \infty)$ to $[0, \infty)$, and give a formula for the inverse function $f^{-1} \colon [0, \infty) \to [1, \infty)$.

   (e) Sketch the graph of f.

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. [**Falkner Section 10 Exercise 28**] Find $\mathscr{P}\big(\{1,2,3\}\big)$.

2. [**Falkner Section 10 Exercise 35**] Let A, B, C, and D be sets. Suppose that $A \times B = C \times D \neq \varnothing$. Prove that $A = C$ and $B = D$.

3. [**Falkner Section 10 Exercise 36**] Let A, B, and C be sets. Prove that

$$A \setminus C \subseteq (A \setminus B) \cup (B \setminus C).$$

4. [**Falkner Section 11 Exercise 7**] Let A and B be sets, and let π_A and π_B be the projections from $A \times B$ to A and B respectively. That is,

$$\begin{aligned} \pi_A\colon A \times B &\to A \\ (x,y) &\mapsto x. \end{aligned} \qquad \begin{aligned} \pi_B\colon A \times B &\to B \\ (x,y) &\mapsto y. \end{aligned}$$

Show that if $B \neq \varnothing$, then $\operatorname{Rng}(\pi_A) = A$, and that if $A \neq \varnothing$, then $\operatorname{Rng}(\pi_B) = B$.