

HOMEWORK 23
MATH 3345 – SPRING 2024 – KUTLER

Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. **[Falkner Section 11 Exercise 20 – modified]** Let

$$\begin{array}{ll} g: [0, 1) \rightarrow \mathbb{R} & h: (-1, 0) \rightarrow \mathbb{R} \\ x \mapsto \frac{x}{1-x} & x \mapsto \frac{x}{1+x} \end{array}$$

- (a) Prove that $\text{Rng}(g) = [0, \infty)$ and $\text{Rng}(h) = (-\infty, 0)$.
 - (b) Prove that both g and h are injections.
 - (c) Conclude that g is a bijection from $[0, 1)$ to $[0, \infty)$ and that h is a bijection from $(-1, 0)$ to $(-\infty, 0)$.
 - (d) Find formulas for $g^{-1}: [0, \infty) \rightarrow [0, 1)$ and $h^{-1}: (-\infty, 0) \rightarrow (-1, 0)$.
2. **[Falkner Section 15 Exercise 1 – modified]** Show that the intervals $A = [1, \infty)$ and $B = (1, \infty)$ have the same cardinality by giving an example of a bijection $f: A \rightarrow B$.
[HINT: Use one simple formula to define f on \mathbb{N} and a different, even simpler formula to define f on $A \setminus \mathbb{N}$.]
Be sure to prove that f is a bijection.

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. Let

$$A = \{n \in \mathbb{N} \mid n \equiv 3 \pmod{4}\}.$$

Define a bijection

$$f: \mathbb{N} \rightarrow A$$

and prove that it is a bijection.

2. **[Falkner Section 11 Exercise 23]** Let

$$\begin{aligned}\varphi: (-1, 1) &\rightarrow \mathbb{R} \\ x &\mapsto \frac{x}{1 - |x|}.\end{aligned}$$

- (a) Show that φ is a bijection from $(-1, 1)$ to \mathbb{R} .
- (b) Find a formula for $\varphi^{-1}: \mathbb{R} \rightarrow (-1, 1)$.

[HINT: Use Exercise 1 above.]

3. **[Falkner Section 15 Exercises 6 & 7 – modified]**

- (a) Show that the intervals $[0, 1)$ and $(0, 1]$ have the same cardinality by giving an example of a bijection $f: [0, 1) \rightarrow (0, 1]$.
- (b) Show that the intervals $(0, 1]$ and $(0, 1)$ have the same cardinality by giving an example of a bijection $g: (0, 1] \rightarrow (0, 1)$.
- (c) Show that the intervals $[0, 1]$ and $[0, 1)$ have the same cardinality by giving an example of a bijection $h: [0, 1] \rightarrow [0, 1)$.
- (d) Conclude that the four intervals $[0, 1]$, $[0, 1)$, $(0, 1]$, and $(0, 1)$ all have the same cardinality.
- (e) Use the functions f , g , and h to construct a bijection from $[0, 1]$ to $(0, 1)$. [HINT: Use Exercise 2 from Homework 22.]