Exam 1 Practice Problems

1. Show that $(P \wedge Q) \vee R$ is logically equivalent to $(P \vee R) \wedge (Q \vee R)$ in two ways

   (a) By using a truth table;

   (b) By giving an explanation in words.

2. Below, you are asked to determine if two sentences are logically equivalent. If the answer is **yes**, provide a proof (either in words or by using a truth table). If the answer is **no**, demonstrate this by choosing appropriate truth values for P, Q, R, and provide a brief justification.

   (a) Is $P \Rightarrow (Q \Rightarrow R)$ logically equivalent to $(P \Rightarrow Q) \Rightarrow R$?

   (b) Is $P \Rightarrow (Q \Rightarrow R)$ logically equivalent to $Q \Rightarrow (P \Rightarrow R)$?

   (c) Is $(P \wedge Q) \Rightarrow R$ logically equivalent to $(P \Rightarrow R) \vee (Q \Rightarrow R)$?

   (d) Is $(P \vee Q) \Rightarrow R$ logically equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$?

   (e) Is $\neg\big[(P \Rightarrow Q) \wedge P\big]$ logically equivalent to $(Q \Rightarrow P) \vee \neg P$?

3. For each sentence below, determine if the sentence is **always true** (i.e., it is a tautology) or if it is **possibly false**. If it is always true, provide a proof (either in words or by using a truth table). If it is possibly false, give truth values for P, Q, R making the sentence false, and provide a brief justification.

   (a) $P \wedge \neg P$

   (b) $P \vee \neg P$

   (c) $(P \wedge Q) \Rightarrow (P \vee Q)$

   (d) $(P \vee Q) \Rightarrow (P \wedge Q)$

   (e) $P \Rightarrow (Q \Rightarrow P)$

   (f) $(P \Rightarrow Q) \Rightarrow P$

   (g) $\big[(P \Rightarrow Q) \wedge \neg Q\big] \Rightarrow \neg P$

   (h) $\big[(P \Rightarrow Q) \wedge (Q \Rightarrow R)\big] \Rightarrow (P \Rightarrow R)$

4. In class, we saw that $\big[(P \Rightarrow Q) \wedge P\big] \Rightarrow Q$ is a tautology, called *modus ponens*.

   (a) Show that $(P \Rightarrow Q) \wedge P$ is logically equivalent to $P \wedge Q$.

   (b) Show that the converse of modus ponens is not a tautology. That is, find truth values for P and Q so that the sentence $Q \Rightarrow \big[(P \Rightarrow Q) \wedge P\big]$ is false.

5. Show that each of the following conditional sentences is a tautology by writing a conditional proof.

   (a) $P \Rightarrow (P \vee Q)$

   (b) $\big[P \Rightarrow (Q \wedge \neg Q)\big] \;\Rightarrow\; \neg P$

   (c) $\big[(P \Rightarrow \neg Q) \wedge (R \Rightarrow Q)\big] \;\Rightarrow\; (P \Rightarrow \neg R)$

   (d) $\big\{\big[(P \Rightarrow Q) \wedge (R \Rightarrow S)\big] \wedge (\neg Q \vee \neg S)\big\} \;\Rightarrow\; (\neg P \vee \neg R)$

6. Below, you are asked to determine if two sentences are logically equivalent. If the answer is **yes**, provide a proof (either in words or by using a truth table). If the answer is **no**, demonstrate this by giving a set A and sentences $P(x)$ and $Q(x)$ making it false, and provide a brief justification.

   (a) Is
   $$(\exists x \in A)\big(P(x) \wedge Q(x)\big)$$
   logically equivalent to
   $$\big((\exists x \in A)P(x)\big) \wedge \big((\exists x \in A)Q(x)\big)?$$

   (b) Is
   $$(\forall x \in A)\big(P(x) \wedge Q(x)\big)$$
   logically equivalent to
   $$\big((\forall x \in A)P(x)\big) \wedge \big((\forall x \in A)Q(x)\big)?$$

   (c) Is
   $$(\exists x \in A)\big(P(x) \Rightarrow Q(x)\big)$$
   logically equivalent to
   $$\big((\exists x \in A)P(x)\big) \Rightarrow \big((\exists x \in A)Q(x)\big)?$$

7. For each of the following sentences, write out what it means in words, state whether it is true or false, and prove your statement.

   (a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 0)$

   (b) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 0)$

   (c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 20)$

   (d) $(\forall x \in \mathbb{R})\big[(x \neq 0) \Rightarrow (\exists y \in \mathbb{R})(xy = 20)\big]$

   (e) $(\forall x \in \mathbb{R})\big[(x \neq 0) \Rightarrow (\exists! y \in \mathbb{R})(xy = 20)\big]$

   (f) $(\forall m \in \mathbb{Z})\big[(m \neq 0) \Rightarrow (\exists! n \in \mathbb{Z})(mn = 20)\big]$

   (g) $(\forall m \in \mathbb{Z})(\exists n \in \mathbb{Z})(m < n)$

   (h) $(\exists n \in \mathbb{Z})(\forall m \in \mathbb{Z})(m < n)$

8. Prove the following statements using mathematical induction. Be sure to clearly state the inductive hypothesis, and explain what you are proving in the inductive step.

   (a) For every $n \in \mathbb{N}$,
   $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

   (b) For every $n \in \mathbb{N}$,
   $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (c) For every $n \in \mathbb{N}$,
   $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

   (d) For every $n \in \mathbb{N}$,
   $$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

   (e) For every $n \in \mathbb{N}$ such that $n > 3$, $n! > 2^n$.

   (f) For every $n \in \mathbb{N}$ such that $n > 6$, $n! > 3^n$.

   (g) Let $x \neq 1$ be a real number. For every $n \in \mathbb{N}$,
   $$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

9. Prove the following.

   (a) The sum of two odd integers is even.

   (b) The sum of an even and an odd integer is odd.

   (c) The sum of two even integers is even.

   (d) The product of two odd integers is odd.

   (e) The product of an even integer and an odd integer is even.

   (f) The product of two even integers is even.

10. Let $n, m \in \mathbb{Z}$. Prove the following.

   (a) If nm is odd, then n is odd and m is odd.

   (b) If nm is even, then n is even or m is even.

   (c) If n^2 is odd, then n is odd.

   (d) If n^2 is even, then n is even.