Exam 2 Practice Problems

1. Let $x, y \in \mathbb{R}$. Prove the following.
   (a) If x and y are rational, then $x + y$ is rational.
   (b) If x and y are rational, then xy is rational.
   (c) If y is rational and $y \neq 0$, then $1/y$ is rational.
   (d) If x and y are rational and $y \neq 0$, then x/y is rational.
   (e) If x is rational and y is irrational, then $x + y$ is irrational.
   (f) If x is rational and y is irrational, then xy is irrational.
   (g) If y is irrational, then $1/y$ is irrational. (Why is $y \neq 0$?)
   (h) If x is rational and y is irrational, then x/y is irrational.

2. Give examples to prove the following statements.
   (a) There exist irrational numbers x and y such that $x + y$ is irrational.
   (b) There exist irrational numbers x and y such that $x + y$ is rational.
   (c) There exist irrational numbers x and y such that xy is irrational.
   (d) There exist irrational numbers x and y such that xy is rational.

3. Prove the following.
   [Hint: Use the fact that any rational number can be written in lowest terms.]
   (a) $\sqrt{2}$ is irrational.
   (b) $\sqrt{3}$ is irrational.
   (c) $\sqrt{6}$ is irrational.
   (d) $\sqrt{2} + \sqrt{3}$ is irrational.

4. Let $d, n \in \mathbb{N}$. Use the definition of divisibility to show that if $d|n$, then $d \leq n$.

5. Let $a, b \in \mathbb{Z}$. Use the definition of divisibility to show that if $a|b$, then $a^2|b^2$.

6. Let a, b, q, r be integers such that $a = bq + r$. Prove that $\gcd(a, b) = \gcd(b, r)$.

7. Let $d \in \mathbb{N}$ and $n \in \mathbb{Z}$. Show that if $d|n$ and $d|(n + 1)$, then $d = 1$.

8. Let P be the sentence

   For all $a, b \in \mathbb{Z}$, if $a|b$ then $a|(b + 5a^2)$.

   Let Q be the sentence

   For all $a, b \in \mathbb{Z}$, if $a|b$ then $b + 5a^2$ is not prime.

   (a) Is the sentence P true? If so, provide a proof. If not, provide a counterexample.

   (b) Is the sentence Q true? If so, provide a proof. If not, provide a counterexample.

9. Use complete induction to prove that every natural number $n \geq 2$ is either prime or a product of primes.

10. Use complete induction to prove the following statement:

    For every $n \in \mathbb{Z}$ such that $n \geq 14$, there exist non-negative integers a and b such that $3a + 7b = n$.

11. Define the **Fibonacci numbers** $F_1, F_2, F_3, \ldots$ by the recurrence relation

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \qquad \text{for all } n \geq 3. \end{aligned}$$

    (a) Compute the first 10 Fibonacci numbers.

    (b) Use (ordinary) induction to prove that for all $n \in \mathbb{N}$,

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

    (c) Use complete induction to prove that for all $n \in \mathbb{N}$, $F_n < (5/3)^n$.

    (d) Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. Use complete induction to prove that for all $n \in \mathbb{N}$, $F_n = \dfrac{a^n - b^n}{a - b}$.

12. Find $\gcd(84, 135)$ in two ways:

    - By using the Euclidean algorithm.
    - By using prime factorization.

    Which way do you prefer?

13. Use the prime factorizations

$$3,219,398 = 2 \cdot 7^3 \cdot 13 \cdot 19^2 \qquad \text{and} \qquad 158,184 = 2^3 \cdot 3^2 \cdot 13^3$$

to find $\gcd(3,219,398\ ,\ 158,184)$. Explain your reasoning.

14. (a) Use the Euclidean algorithm to compute $\gcd(30, 72)$.
    (b) Find integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 6$.
    (c) Do there exist integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 18$?
    (d) Do there exist integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 15$?

15. Find integers x and y such that $162x + 31y = 1$.

16. (a) Let $a \in \mathbb{N}$ and let p be a prime number. Prove that if p does not divide a, then $\gcd(p, a) = 1$.
    (b) Show that there exists $a \in \mathbb{N}$ such that 12 does not divide a and $\gcd(12, a) \neq 1$.

17. Let $a \in \mathbb{N}$ and let p be a prime number. Prove that if $p|a^2$, then $p|a$.

    [HINT: Use unique prime factorization.]

18. Let n be an even integer. Prove that there exist unique integers $q, r \in \mathbb{Z}$ such that

$$n = 6q + r$$

and $r \in \{0, 2, 4\}$.

19. Let $m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$. Prove that if

$$a \equiv b \mod m \qquad \text{and} \qquad c \equiv d \mod m,$$

then

$$a - c \equiv b - d \mod m.$$

20. Without using a calculator, find the natural number k such that $0 \leq k \leq 14$ and k satisfies the given congruence.
    (a) $2^{75} \equiv k \pmod{15}$
    (b) $6^{41} \equiv k \pmod{15}$
    (c) $140^{874} \equiv k \pmod{15}$

21. Without using a calculator, show that 15 divides $37^{42} - 38^{90}$.

22. Prove that
$$7^n \equiv 1 + 6n \mod 9$$
for every $n \in \mathbb{N}$.

23. Let A and B be the following sets.
$$A = \{n \in \mathbb{Z} \mid \text{there exists } k \in \mathbb{Z} \text{ such that } n = 4k + 2\}$$
$$B = \{n \in \mathbb{Z} \mid n \text{ is even}\}.$$
   (a) Prove that $A \subseteq B$.
   (b) Pove that $B \not\subseteq A$.

24. Let
$$A = \{x \in \mathbb{R} \mid x^2 \in \mathbb{Q}\}.$$
   (a) Prove that $\mathbb{Q} \subseteq A$.
   (b) Explain why $\mathbb{Q} \neq A$.

25. Let A and B be sets. Prove the following equalities of sets. (Recall that to prove two sets are equal, we must show one is a subset of the other and vice versa.)
   (a) $A \setminus (A \setminus B) = A \cap B$.
   (b) $(A \cup B) \setminus B = A \setminus B$.
   (c) $(A \cap B) \setminus B = \varnothing$.
   (d) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

26. Let A and B be sets. Prove the following statements.
   (a) $A \subseteq B$ if and only if $A \cap B = A$
   (b) $A \subseteq B$ if and only if $A \setminus B = \varnothing$.
   (c) $A \subseteq B$ if and only if $A \cup B = B$.