Final Exam Practice Problems

The Final Exam will be cumulative. The practice problems below primarily relate to the topics we have covered since Exam 2. Please also review the material covered on the midterm exams.

1. (a) For each $n \in \mathbb{N}$, let $A_n = [n-1, n]$. Prove the following:
      i. $\bigcup_{n=1}^{\infty} A_n$ is equal to $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$.
      ii. $\bigcap_{n=1}^{\infty} A_n$ is equal to $\emptyset$.

   (b) For each $n \in \mathbb{N}$, let $B_n = (n-1, n)$. Prove the following:
      i. $\bigcup_{n=1}^{\infty} B_n$ is equal to $\mathbb{R}_{\geq 0} \setminus \mathbb{Z}$.
      ii. $\bigcap_{n=1}^{\infty} B_n$ is equal to $\emptyset$.

   (c) For each $n \in \mathbb{N}$, let $C_n = (-n, n)$. Prove the following:
      i. $\bigcup_{n=1}^{\infty} C_n$ is equal to $\mathbb{R}$
      ii. $\bigcap_{n=1}^{\infty} C_n$ is equal to $(-1, 1)$.

2. Let A, B, C, and D be sets. Prove the following:

   (a) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

   (b) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

3. (a) Let A be a set. Prove that $A \times \emptyset = \emptyset$.

   (b) Let A, B, and C be sets. Prove that if $C \neq \emptyset$ and $A \times C = B \times C$, then $A = B$.

4. Let A and B be subsets of the real numbers. Define a function $f\colon A \to B$ by the formula $f(x) = x^2$. Which of the following choices of A and B are allowable?

   (a) $A = \mathbb{R}$ and $B = \mathbb{R}$.

   (b) $A = \mathbb{R}$ and $B = [0, \infty)$.

   (c) $A = [0, \infty)$ and $B = \mathbb{R}$.

   (d) $A = [0, \infty)$ and $B = [0, \infty)$.

   (e) $A = (-\infty, 0]$ and $B = [0, \infty)$.

   (f) $A = (-\infty, 0]$ and $B = (-\infty, 0]$.

   (g) $A = [0, 2]$ and $B = [0, 4]$.

   (h) $A = [-2, 2]$ and $B = [0, 4]$.

   (i) $A = [-2, 3]$ and $B = [0, 4]$.

5. Let $A = \{1, 2, 3\}$ and $B = \{3, 6, 9, 12, 15\}$. In each case below, give an example of the requested type of function, or prove that no such function exists.

   (a) A surjective function $f: A \to B$.

   (b) An injective function $g: A \to B$.

   (c) A surjective function $h: B \to A$.

   (d) An injective function $k: B \to A$.

6. Let $A = \mathbb{N}$ and $B = \mathbb{Z}$. In each case below, give an example of the requested type of function, or prove that no such function exists.

   (a) A surjective function $f: A \to B$.

   (b) An injective function $g: A \to B$.

   (c) A surjective function $h: B \to A$.

   (d) An injective function $k: B \to A$.

7. Let $f: \mathbb{R} \to \mathbb{R}$ be a function.

   - We say that f is **strictly increasing** if for all $x_1, x_2 \in \mathbb{R}$, if $x_1 < x_2$ then $f(x_1) < f(x_2)$.
   - We say that f is **strictly decreasing** if for all $x_1, x_2 \in \mathbb{R}$, if $x_1 < x_2$ then $f(x_1) > f(x_2)$.

   (a) Prove that if $f: \mathbb{R} \to \mathbb{R}$ is strictly increasing, then f is injective.

   (b) Prove that if $f: \mathbb{R} \to \mathbb{R}$ is strictly decreasing, then f is injective.

   (c) Give an example of a strictly increasing or strictly decreasing function from $\mathbb{R}$ to $\mathbb{R}$ which is **not** surjective.

8. (a) Give an example of a function $f: \mathbb{N} \to \mathbb{N}$ which is injective but not surjective.

   (b) Give an example of a function $g: \mathbb{N} \to \mathbb{N}$ which is surjective but not injective.

9. (a) Give an example of a function $f: [0, 4] \to [0, 4]$ which is injective but not surjective.

   (b) Give an example of a function $g: [0, 4] \to [0, 4]$ which is surjective but not injective.

   (c) Give an example of a function $g: [0, 4] \to [0, 4]$ which is a bijection, but is not strictly increasing or strictly decreasing.

10. Let $f\colon A \to B$ and $g\colon B \to C$. Consider the composition $g \circ f\colon A \to C$.

   (a) Prove that if f is an injection and g is an injection, then $g \circ f$ is an injection.

   (b) Prove that if f is a surjection and g is a surjection, then $g \circ f$ is a surjection.

   (c) Prove that if f is a bijection and g is a bijection, then $g \circ f$ is a bijection.

   (d) Suppose $g \circ f$ is a bijection. Does it follow that f and g are bijections?

11. Let $S = \mathbb{Z} \times \{0, 1, 2, 3, 4, 5\}$. Define a function $g\colon S \to \mathbb{Z}$ by

$$g(n, i) = 6n + i.$$

   (a) Prove or disprove: g is injective.

   (b) Prove or disprove: g is surjective.

12. Let $f\colon A \to B$ be a bijection. Use the definitions of injective and surjective functions to prove the following:

   For each $y \in B$, there exists a unique $x \in A$ such that $f(x) = y$.

13. Let $f\colon A \to B$ be a bijection.

   (a) Give the definition of the inverse function $f^{-1}\colon B \to A$.

   (b) Prove that f^{-1} is also a bijection.

14. (a) Let A be the set of even positive integers. Define a bijection $f\colon \mathbb{N} \to A$ and prove that it is a bijection.

   (b) Let B be the set of odd positive integers. Define a bijection $f\colon \mathbb{N} \to B$ and prove that it is a bijection.

   (c) Let C be the set of positive integers which are divisible by 7. Define a bijection $f\colon \mathbb{N} \to C$ and prove that it is a bijection.

   (d) Let D be the set of positive integers which leave a remainder of 4 when divisided by 7. Define a bijection $f\colon \mathbb{N} \to D$ and prove that it is a bijection.

15. Let A and B be finite sets, and let $f\colon A \to B$ be a function.

   (a) Prove that if f is injective, then $|A| = |\operatorname{Rng}(f)|$. Deduce that $|A| \leq |B|$.

   (b) Prove that if f is surjective, then $|A| \geq |B|$.

16. Let A and B be finite sets.

    (a) Prove that
    $$|A \times B| = |A| \cdot |B|.$$

    (b) Is it correct to say that $A \times B = B \times A$?

    (c) Define a bijection $f\colon A \times B \to B \times A$, and prove that it is a bijection.

17. Let A be a finite set. Prove that if $|A| = n$, then $|\mathscr{P}(A)| = 2^n$.

18. Define a bijection $f\colon \mathbb{N} \to \mathbb{Z}$. Prove that f is a bijection.

19. Define a bijection $g\colon \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$. Prove that g is a bijection.

20. Let $a, b \in \mathbb{R}$ with $a < b$. Define a bijection $h\colon (0,1) \to (a,b)$.

21. Let $\varphi\colon (-1,1) \to \mathbb{R}$ be defined by
    $$\varphi(x) = \frac{x}{1-|x|}.$$

    (a) Prove that φ is a bijection.

    (b) Find a formula for φ^{-1}.
    [HINT: Use the piecewise formula
    $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$
    to write φ^{-1} as a piecewise function. Can you find a unified (i.e., non-piecewise) formula for φ^{-1}?]