MATH 3345 BONUS PROBLEMS #1

The following bonus problems are worth extra credit, which will be **holistically** incorporated into your final grade computation.

These bonus problems are not a substitute for the ordinary homework assignments. Rather, you should view these problems as an optional insurance policy on your final course grade .

You may turn in any number of these problems, individually or in batches, at any time but **no later than Monday, November 15.**

1. We define a logical connective $\downarrow$ as follows: $P \downarrow Q$ is true when both P and Q are false, and it is false otherwise. (We read $P \downarrow Q$ as "P nor Q").

   (a) Write a truth table for $P \downarrow Q$ and check that $P \downarrow Q$ is logically equivalent to $\neg(P \vee Q)$.

   (b) Check that $P \downarrow Q \equiv Q \downarrow P$. That is, $\downarrow$ is commutative.

   (c) Show that $(P \downarrow Q) \downarrow R$ and $P \downarrow (Q \downarrow R)$ are logically inequivalent. That is, $\downarrow$ is *not* associative.

   (d) Show that the logical connectives $\neg$, $\wedge$, and $\vee$ can each be expressed in terms of $\downarrow$ without using any other logical connectives. Specifically, prove the following:

      i. $\neg P \equiv (P \downarrow P)$.

      ii. $P \wedge Q \equiv (P \downarrow P) \downarrow (Q \downarrow Q)$.

      iii. $P \vee Q \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$.

   (e) Prove that the logical connective $\Rightarrow$ can be expressed in terms of $\downarrow$ without using any other logical connectives.

2. Let a be an odd integer. Prove by induction that $a^{2^n} - 1$ is divisible by 2^{n+1} for every integer $n \geq 0$.

3. Let P denote the following sentence:

   Let $n, d, p \in \mathbb{Z}$ be integers such that $d > 0$ and p is prime. If d divides n and d divides $n + p$, then $d = 1$ or p divides n.

   (a) Write P as a logical sentence using quantifiers ($\forall$, $\exists$) and logical connectives ($\wedge$, $\vee$, $\Rightarrow$, etc.).

   (b) Write the negation $\neg P$.

   (c) Which statement is true, P or $\neg P$? Prove the true statement.

4. For $n, k \in \mathbb{Z}$ such that $0 \leq k \leq n$, define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(a) Prove that $\binom{n}{0} = 1$, $\binom{n}{n} = 1$, and

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \text{if} \qquad 1 \leq k \leq n-1.$$

(b) Use part (a) and induction to prove that $\binom{n}{k}$ is a positive integer for all $n, k \in \mathbb{Z}$ such that $0 \leq k \leq n$.

(c) Let $x, y \in \mathbb{R}$. Prove that for every integer $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

5. The **Archimedean Property** of the real numbers is the following statement:

> For every $x, y \in \mathbb{R}$ such that $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.

(a) Show that, for every $a \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > a$. That is, $\mathbb{N}$ is not bounded above. [HINT: Proceed by contradiction, and use the Least Upper Bound Property of $\mathbb{R}$.]

(b) Use part (a) to prove that the Archimedean Property is true.

(c) Use the Archimedean Property to prove that for every $x \in \mathbb{R}$ such that $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

(d) Prove that there is a rational number between any two real numbers. That is, for every $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$. [HINT: Start by using part (c) to find a denominator for q. Then, use the Well-Ordering Axiom to choose a numerator for q.]

6. Let $a, b, c \in \mathbb{Z}$ be integers with a and b not both 0. Let $d = \gcd(a, b)$.

(a) Prove that there exist $x, y \in \mathbb{Z}$ such that

$$ax + by = c$$

if and only if d divides c.

(b) Suppose there exist $x_0, y_0 \in \mathbb{Z}$ such that

$$ax_0 + by_0 = c.$$

Show that for every $k \in \mathbb{Z}$, the numbers

$$x = x_0 + \frac{kb}{d} \qquad \text{and} \qquad y = y_0 - \frac{ka}{d}$$

are integers and $ax + by = c$.

(c) Suppose still that $x_0, y_0 \in \mathbb{Z}$ satisfy

$$ax_0 + by_0 = c.$$

Show that if $x, y \in \mathbb{Z}$ satisfies the equation $ax + by = c$, then

$$x = x_0 + \frac{kb}{d} \qquad \text{and} \qquad y = y_0 - \frac{ka}{d}$$

for some $k \in \mathbb{Z}$.

(d) Use the results from parts (a)–(c) to explain why the equation

$$18x + 42y = 30$$

has integer solutions, and find all integer solutions $x, y \in \mathbb{Z}$.