Math 3345 Bonus Problems #2

The following bonus problems are worth extra credit, which will be **holistically** incorporated into your final grade computation.

These bonus problems are not a substitute for the ordinary homework assignments. Rather, you should view these problems as an optional insurance policy on your final course grade .

You may turn in any number of these problems, individually or in batches, at any time but **no later than Thursday, December 16.** Please submit them in to my office (MW 756) or by email in a **single .pdf file**.

1. For sets A and B, define
$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A).$$
   (a) Let A and B be sets. Prove that $A = B$ if and only if $A \,\Delta\, B = \emptyset$.
   (b) Let A and B be sets. Prove that $A \,\Delta\, B = B \,\Delta\, A$.
   (c) Let A, B, and C be sets. Prove that $(A \,\Delta\, B) \,\Delta\, C = A \,\Delta\, (B \,\Delta\, C)$.

2. For each $n \in \mathbb{N}$, define $A_n = [0, 1 + \frac{1}{n}]$ and $B_n = [0, 1 - \frac{1}{n}]$.
   (a) Prove that $\bigcup_{n=1}^{\infty} A_n$ is an interval, and describe this interval explicitly.
   (b) Prove that $\bigcap_{n=1}^{\infty} A_n$ is an interval, and describe this interval explicitly.
   (c) Prove that $\bigcup_{n=1}^{\infty} B_n$ is an interval, and describe this interval explicitly.
   (d) Prove that $\bigcap_{n=1}^{\infty} B_n$ is an interval, and describe this interval explicitly.

3. Define a function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by
$$f(m, n) = (5m + 4n,\ 4m + 3n).$$
   Prove that f is a bijection and give a formula for its inverse function.

4. (a) Find a bijection $f\colon (0, \pi) \to \mathbb{R}$ or prove that no such bijection exists.
   (b) Find a bijection $g\colon [0, \pi] \to \mathbb{R}$ or prove that no such bijection exists.

5. Let A be an infinite set. Prove that there exists an injective function $f\colon \mathbb{N} \to A$.