Math 3345 Bonus Problems

The following bonus problems are worth extra credit, which will be added to the homework component of your course grade (not to exceed 100%).

Each problem is worth up to 5 points. Full credit will only be awarded to solutions which are **complete, correct, and readable.**

You may turn in solutions to any number of these problems, individually or in batches, **at any time but no later than Monday, November 28.**

1. We define a logical connective $\downarrow$ as follows: $P \downarrow Q$ is true when both P and Q are false, and it is false otherwise. (We read $P \downarrow Q$ as "P nor Q").

   (a) Write a truth table for $P \downarrow Q$ and check that $P \downarrow Q$ is logically equivalent to $\neg(P \vee Q)$.

   (b) Check that $P \downarrow Q \equiv Q \downarrow P$. That is, $\downarrow$ is commutative.

   (c) Show that $(P \downarrow Q) \downarrow R$ and $P \downarrow (Q \downarrow R)$ are logically inequivalent. That is, $\downarrow$ is *not* associative.

   (d) Show that the logical connectives $\neg$, $\wedge$, and $\vee$ can each be expressed entirely in terms of $\downarrow$, without using any other logical connectives. Specifically, prove the following:

      i. $\neg P \equiv (P \downarrow P)$.

      ii. $P \wedge Q \equiv (P \downarrow P) \downarrow (Q \downarrow Q)$.

      iii. $P \vee Q \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$.

   (e) Prove that the logical connective $\Rightarrow$ can be expressed entirely in terms of $\downarrow$. That is, show that the sentence $P \Rightarrow Q$ is logically equivalent to a sentence involving $\downarrow$ and no other logical connectives.

2. Provide three proofs of the logical equivalence $P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$:

   (a) Prove the logical equivalence using a truth table.

   (b) Prove the logical equivalence using a symbolic argument. [HINT: Start with the right-hand side $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ and expand it using two applications of the Distributive Laws.]

   (c) Prove the logical equivalence using an explanation in words.

3. Let a be an odd integer. Prove by induction that $a^{2^n} - 1$ is divisible by 2^{n+1} for every integer $n \geq 0$.

4. Let P denote the following sentence:

   Let $n, d, p \in \mathbb{Z}$ be integers such that $d > 0$ and p is prime. If d divides n and d divides $n + p$, then $d = 1$ or p divides n.

   (a) Write P as a logical sentence using quantifiers ($\forall$, $\exists$) and logical connectives ($\wedge$, $\vee$, $\Rightarrow$, etc.).

   (b) Write the negation $\neg P$.

   (c) Which statement is true, P or $\neg P$? Prove the true statement.

5. For $n, k \in \mathbb{Z}$ such that $0 \le k \le n$, define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

   Note that, by convention, $0! = 1$.

   (a) Prove that $\binom{n}{0} = 1$, $\binom{n}{n} = 1$, and

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \text{if} \qquad 1 \le k \le n-1.$$

   (b) Use part (a) and induction to prove that $\binom{n}{k}$ is a positive integer for all $n, k \in \mathbb{Z}$ such that $0 \le k \le n$.

   (c) Let $x, y \in \mathbb{R}$. Prove that for every integer $n \ge 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

6. The **Archimedean Property** of the real numbers is the following statement:

   For every $x, y \in \mathbb{R}$ such that $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.

   (a) Prove that for every $a \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > a$. That is, $\mathbb{N}$ is not bounded above. [HINT: Proceed by contradiction, and use the Least Upper Bound Property of $\mathbb{R}$.]

   (b) Use part (a) to prove that the Archimedean Property is true.

   (c) Use the Archimedean Property to prove that for every $x \in \mathbb{R}$ such that $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

   (d) Prove that there is a rational number between any two real numbers. That is, for every $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$. [HINT: Start by using part (c) to find a denominator for q. Then, use the Well-Ordering Axiom to choose a numerator for q.]

7. Let $a, b, c \in \mathbb{Z}$ be integers with a and b not both 0. Let $d = \gcd(a, b)$.

   (a) Prove that there exist $x, y \in \mathbb{Z}$ such that
   $$ax + by = c$$
   if and only if d divides c.

   (b) Suppose there exist $x_0, y_0 \in \mathbb{Z}$ such that
   $$ax_0 + by_0 = c.$$
   Show that for every $k \in \mathbb{Z}$, the numbers
   $$x = x_0 + \frac{kb}{d} \qquad \text{and} \qquad y = y_0 - \frac{ka}{d}$$
   are integers and $ax + by = c$.

   (c) Suppose still that $x_0, y_0 \in \mathbb{Z}$ satisfy
   $$ax_0 + by_0 = c.$$
   Show that if $x, y \in \mathbb{Z}$ satisfies the equation $ax + by = c$, then
   $$x = x_0 + \frac{kb}{d} \qquad \text{and} \qquad y = y_0 - \frac{ka}{d}$$
   for some $k \in \mathbb{Z}$.

   (d) Use the results from parts (a)–(c) to explain why the equation
   $$18x + 42y = 30$$
   has integer solutions, and find all integer solutions $x, y \in \mathbb{Z}$.

8. Let p be an integer such that $p \geq 2$. Suppose that for all $x, y \in \mathbb{Z}$, if $p|xy$ then $p|x$ or $p|y$. Prove that p is prime. (This is the converse of the "Theorem on Division by a Prime.")

9. (a) Let $x \in \mathbb{Q}$. Prove that if $x^3 \in \mathbb{Z}$, then $x \in \mathbb{Z}$.

   (b) Let $n \in \mathbb{Z}$. Prove that if n is not a perfect cube (i.e., there is no integer m such that $n = m^3$), then $\sqrt[3]{n}$ is irrational.

10. Let $c, n \in \mathbb{N}$. Use the rational roots theorem (see Homework 17) to prove that $\sqrt[n]{c}$ is either an integer or an irrational number. [HINT: Consider the polynomial $x^n - c$.]

11. For sets A and B, define

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A).$$

(a) Let A and B be sets. Prove that $A = B$ if and only if $A \,\Delta\, B = \emptyset$.

(b) Let A and B be sets. Prove that $A \,\Delta\, B = B \,\Delta\, A$.

(c) Let A, B, and C be sets. Prove that $(A \,\Delta\, B) \,\Delta\, C = A \,\Delta\, (B \,\Delta\, C)$.

12. For each $n \in \mathbb{N}$, define $A_n = [0, 1 + \frac{1}{n}]$ and $B_n = [0, 1 - \frac{1}{n}]$.

(a) Prove that $\bigcup_{n=1}^{\infty} A_n$ is an interval, and describe this interval explicitly.

(b) Prove that $\bigcap_{n=1}^{\infty} A_n$ is an interval, and describe this interval explicitly.

(c) Prove that $\bigcup_{n=1}^{\infty} B_n$ is an interval, and describe this interval explicitly.

(d) Prove that $\bigcap_{n=1}^{\infty} B_n$ is an interval, and describe this interval explicitly.

13. Define a function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by

$$f(m, n) = (5m + 4n,\ 4m + 3n).$$

Prove that f is a bijection and give a formula for its inverse function.

14. (a) Find a bijection $f\colon (0, \pi) \to \mathbb{R}$ or prove that no such bijection exists.

(b) Find a bijection $g\colon [0, \pi] \to \mathbb{R}$ or prove that no such bijection exists.

15. Let A be an infinite set. Prove that there exists an injective function $f\colon \mathbb{N} \to A$.