Homework 12

Math 3345 – Autumn 2022 – Kutler

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. (a) Let n be an integer with $n > 1$. Prove that n is prime if and only if for every prime p such that $p^2 \leq n$, p does not divide n.
   (b) Use part (a) to prove that 29 is prime.
   (c) Use part (a) to prove that 101 is prime.

2. Let $a, b \in \mathbb{N}$. We say that a positive integer $m \in \mathbb{N}$ is a **common multiple** of a and b if $a|m$ and $b|m$.
   (a) Show that for any $a, b \in \mathbb{N}$, ab is a common multiple of a and b.
   (b) Prove that for any $a, b \in \mathbb{N}$, there exists a common multiple ℓ of a and b such that $\ell \leq m$ if m is any common multiple of a and b. This number ℓ is called the **least common multiple** of a and b. We write $\ell = \text{lcm}(a, b)$.
   (c) Give an example of positive integers $a, b \in \mathbb{N}$ such that $\text{lcm}(a, b) = ab$.
   (d) Give an example of positive integers $a, b \in \mathbb{N}$ such that $\text{lcm}(a, b) < ab$.
   (e) Explain why there do not exist positive integers a and b such that $\text{lcm}(a, b) > ab$.

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. Let n be an integer. Prove that if $3|n^2$, then $3|n$.

2. Find an integer n such that $4|n^2$ but $4 \nmid n$.