Homework 14

Math 3345 – Autumn 2022 – Kutler

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. **[Falkner Section 4 Exercise 10 – modified]** Let x be a rational number and let y be an irrational number. Prove the following statements.
   (a) $-y$ is irrational.
   (b) $x - y$ is irrational.
   (c) $y - x$ is irrational.
   (d) If $x \neq 0$, then xy is irrational. [Be sure to explain where you use the condition that $x \neq 0$ in your proof.]
   (e) Explain why the condition that $x \neq 0$ was necessary for part (d). That is explain why xy is rational when $x = 0$.
   (f) $1/y$ is irrational. [You should explain why $y \neq 0$ **must** be true.]
   (g) If $x \neq 0$, then x/y is irrational.
   (h) If $x \neq 0$, then y/x is irrational.
2. Let $a, b, q, r \in \mathbb{Z}$ such that $a = bq + r$.
   (a) Let $d \in \mathbb{N}$. Prove that d is a common divisor of a and b if and only if d is a common divisor of b and r.
   (b) Use part (a) to conclude that $\gcd(a, b) = \gcd(b, r)$.
3. Use the Euclidean algorithm to compute the following.
   (a) $\gcd(36, 22)$
   (b) $\gcd(96, 112)$
   (c) $\gcd(162, 31)$
   (d) $\gcd(-15, 45)$

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. [**Falkner Section 4 Exercise 16**] Let $n \in \mathbb{N}$. Prove that there exists a prime number q such that $n < q \leq 1 + n!$. [HINT: Take q to be any prime which divides $1 + n!$. (How do we know such a prime exists?) Now explain why $q \leq 1 + n!$ and $q > n$ must both be true.]

2. Let $a, b \in \mathbb{N}$. Prove that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.