Homework 16

Math 3345 – Autumn 2022 – Kutler

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. [**Falkner Section 4 Exercise 25**] Let $m \in \mathbb{N}$. Show that
   (a) For all $a \in \mathbb{Z}$, we have $a \equiv a \mod m$. [**Reflexivity**]
   (b) For all $a, b \in \mathbb{Z}$, if $a \equiv b \mod m$, then $b \equiv a \mod m$. [**Symmetry**]
   (c) For all $a, b, c \in \mathbb{Z}$, if $a \equiv b \mod m$ and $b \equiv c \mod m$, then $a \equiv c \mod m$. [**Transitivity**]

2. [**Falkner Section 4 Exercise 26 – modified**] Let $m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$. Suppose that $a \equiv b \mod m$ and $c \equiv d \mod m$.
   (a) Prove that $a + c \equiv b + d \mod m$.
   (b) Prove that $a - c \equiv b - d \mod m$.
   (c) Prove that $ac \equiv bd \mod m$. [HINT: Since $a \equiv b \mod m$, m divides $b - a$, so $b - a = mk$ for some integer k. Rewrite this as $b = a + mk$. Similarly, $d = c + m\ell$ for some integer ℓ.]

3. Without using a calculator, find the natural number k such that $0 \leq k \leq 14$ and k satisfies the given congruence.
   (a) $2^{75} \equiv k \pmod{15}$
   (b) $6^{41} \equiv k \pmod{15}$
   (c) $140^{874} \equiv k \pmod{15}$

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. Recall that any positive integer $n \in \mathbb{N}$ has a unique **base-10 expression:**

$$n = \sum_{i=0}^{k} a_i \, 10^i,$$

   where $k \geq 0$ and $0 \leq a_i < 0$ for all i. The integers a_i are the **digits** of n. For example,

$$4592 = 2 \cdot 10^0 + 9 \cdot 10^1 + 5 \cdot 10^2 + 4 \cdot 10^4.$$

   Prove the following:

   (a) $2|n$ if and only if 2 divides the "ones digit" a_0.

   (b) $3|n$ if and only if 3 divides the sum of the digits $\sum_{i=0}^{k} a_i$.

   (c) $5|n$ if and only if the ones digit a_0 is equal to 0 or 5.

2. Formulate and prove divisibility criteria similar to those in the previous problem for the conditions

   (a) $4|n$.

   (b) $9|n$.

   (c) $11|n$.