

**HOMEWORK 17**  
MATH 3345 – AUTUMN 2022 – KUTLER

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. Let  $a, b, c \in \mathbb{N}$ . Prove that if  $\gcd(a, b) = 1$  and  $\gcd(a, c) = 1$ , then  $\gcd(a, bc) = 1$ .  
[HINT: First check that the statement is true if any of  $a, b$ , or  $c$  is equal to 1. Then, for the case where  $a > 1, b > 1$ , and  $c > 1$ , consider unique prime factorizations.]
2. Let  $a, b \in \mathbb{N}$  and set  $d = \gcd(a, b)$ .
  - (a) Explain why  $\frac{a}{d}$  and  $\frac{b}{d}$  are integers.
  - (b) Prove that  $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ .  
[HINT: As in the previous problem, there is very little to prove when  $a = 1$  or  $b = 1$ . In the case where  $a > 1$  and  $b > 1$ , look at prime factorizations.]

**Practice Problems**

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. [**Falkner Section 4 Exercise 20 (The rational roots theorem)**] Let  $r \in \mathbb{Q}$  be a rational number such that

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0,$$

where  $n \in \mathbb{N}$  and  $c_0, c_1, \dots, c_n, c_{n-1} \in \mathbb{Z}$ , and  $c_n \neq 0$ . In other words,  $r$  is a rational root of the polynomial  $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ , and every coefficient of this polynomial is an integer.

Prove that  $r$  can be written in the form  $r = \frac{a}{b}$ , where  $a$  is an integer such that  $a|c_0$  and  $b$  is a nonzero integer such that  $b|c_n$ .

[HINT: Write  $r = \frac{a}{b}$  as a fraction in lowest terms. Then  $\gcd(a, b) = 1$ . (Why?) What does this say about the prime factorizations of  $a$  and  $b$ ?]

2. [**Falkner Section 4 Exercise 21**] Let  $f(x) = 3x^3 - 40x^2 + 97x + 10$ .
  - (a) Find a rational number  $r \in \mathbb{Q}$  such that  $f(r) = 0$ . [HINT: Use the rational roots theorem to narrow down the possibilities for  $r$ .]

- (b) Find two other real numbers  $s$  and  $t$  such that  $f(s) = 0$  and  $f(t) = 0$ . [HINT: Use part (a) and polynomial long division to write  $f(x) = (x - r)g(x)$ , where  $g(x)$  is a quadratic polynomial.]
- (c) Explain why  $s$  and  $t$  must be irrational. [HINT: There are several ways to do this. One elegant way is to notice that  $g(x) = 3h(x)$ , where  $h(x)$  is a quadratic polynomial to which it is particularly easy to apply the rational roots theorem.]