Homework 18
Math 3345 – Autumn 2022 – Kutler

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. Let $a, b \in \mathbb{N}$, and suppose that $a^2 = 24b^2$ (equivalently, $\left(\frac{a}{b}\right)^2 = 24$).

   (a) Show that $a \neq 1$.

   (b) From part (a), we conclude that $a \geq 2$, so a has a unique prime factorization. Write it as
   $$a = p_1^{e_1} \cdots p_k^{e_k},$$
   where $p_1, \ldots, p_k$ are distinct primes (i.e., $p_i \neq p_j$ when $i \neq j$) and the exponents e_i are positive integers. What is the unique prime factorization of a^2?

   (c) Describe the unique prime factorization of $24b^2$.
   [HINT: You will need to consider two cases. If $b = 1$, then $24b^2 = 24$. Otherwise, $b \geq 2$ has a unique prime factorization. In this case, you will need to use the prime factorization of b to describe the prime factorization of $24b^2$, similar to what you did in part (b).]

   (d) Use the equality $a^2 = 24b^2$ and your prime factorizations from parts (b) and (c) to derive a contradiction. Conclude that no such $a, b \in \mathbb{N}$ exist.

2. [**Falkner Section 10 Exercise 1**] Which of the sets A, B, C, D, and E below are the same?

   $$A = \{3\}, \qquad B = \{2, 4\}, \qquad C = \{x \mid x \text{ is prime}, x \text{ is odd, and } x < 5\},$$
   $$D = \{x - 1 \mid x \text{ is prime}, x \text{ is odd, and } x < 5\}, \qquad E = \{x^2 + x \mid x \in \{-1, 1\}\}.$$

   Also, how many distinct sets are named here?

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. [**Falkner Section 10 Exercise 4**] Which of the following set notations denote the empty set?

   (a) $\{z \mid z \text{ is a horse and } z \text{ has 6 legs}\}$.

   (b) $\{a \in \mathbb{R} \mid a^2 + 2a + 2 = 0\}$.

   (c) $\{n \in \mathbb{N} \mid n^2 + n + 11 \text{ is not prime}\}$.