

HOMEWORK 22
MATH 3345 – AUTUMN 2022 – KUTLER

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. **[Falkner Section 11 Exercise 6]** Let $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$, let $g(y) = \sqrt{y - 1}$ for all $y \in [1, \infty)$, and let $h(u) = 1 - u$ for all $u \in [2, 3)$. Find the range of f , the range of g , and the range of h .

2. **[Falkner Section 11 Exercise 15(a) – modified]** Recall that for a set X , the **power set** $\mathcal{P}(X)$ is the set of all subsets of X .

Let S and T be sets.

- (a) Prove that if $A \subseteq S$ and $B \subseteq T$, then $A \cup B \subseteq S \cup T$.
- (b) Prove that *every* subset of $S \cup T$ is of the form $A \cup B$, where $A \subseteq S$ and $B \subseteq T$. That is, if $Y \subseteq S \cup T$, then there exist subsets $A \subseteq S$ and $B \subseteq T$ such that $Y = A \cup B$.

We may understand the result of part (a) as saying that the function

$$\begin{aligned} f: \mathcal{P}(S) \times \mathcal{P}(T) &\rightarrow \mathcal{P}(S \cup T) \\ (A, B) &\mapsto A \cup B \end{aligned}$$

is well-defined. That is, if $A \subseteq S$ and $B \subseteq T$, then $f(A, B) = A \cup B$ is a well-defined subset of $S \cup T$.

The result of part (b) then shows that the range of f is all of $\mathcal{P}(S \cup T)$. That is, if $Y \in \mathcal{P}(S \cup T)$, then there exist $A \in \mathcal{P}(S)$ and $B \in \mathcal{P}(T)$ such that $f(A, B) = Y$.

- (c) Illustrate this line of thinking in the case where $S = \{1, 2\}$ and $T = \{2, 3\}$. The eight subsets of $S \cup T = \{1, 2, 3\}$ are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

For each set Y in this list, find $A \subseteq \{1, 2\}$ and $B \subseteq \{2, 3\}$ such that $A \cup B = Y$.

- (d) Continuing with the example from part (c), for which of the sets Y is there a **unique** choice of $A \subseteq \{1, 2\}$ and $B \subseteq \{2, 3\}$ such that $A \cup B = Y$?

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. **[Falkner Section 10 Exercise 28]** Find $\mathcal{P}(\{1, 2, 3\})$.
2. **[Falkner Section 10 Exercise 35]** Let A , B , C , and D be sets. Suppose that $A \times B = C \times D \neq \emptyset$. Prove that $A = C$ and $B = D$.
3. **[Falkner Section 10 Exercise 36]** Let A , B , and C be sets. Prove that

$$A \setminus C \subseteq (A \setminus B) \cup (B \setminus C).$$