Homework 23
Math 3345 – Autumn 2022 – Kutler

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. [**Falkner Section 11 Exercise 15(b) – modified**] Let S and T be sets. Prove that S and T are disjoint (i.e., $S \cap T = \varnothing$) if and only if the following condition holds:

   For all subsets $A_1, A_2 \subseteq S$ and $B_1, B_2 \subseteq T$, if $A_1 \cup B_1 = A_2 \cup B_2$, then $A_1 = A_2$ and $B_1 = B_2$. $(\star)$

   Note: The condition $(\star)$ is equivalent to the statement that the function

   $$\begin{aligned} f\colon \mathscr{P}(S) \times \mathscr{P}(T) &\to \mathscr{P}(S \cup T) \\ (A, B) &\mapsto A \cup B \end{aligned}$$

   is injective. You proved on Homework 22 that this function is always surjective.

2. [**Falkner Section 11 Exercise 17 – modified**] Let

   $$\begin{aligned} f\colon [1, \infty) &\to \mathbb{R} \\ x &\mapsto x - 1. \end{aligned}$$

   (a) Show that $\text{Rng}(f) \subseteq [0, \infty)$. That is, $f(x) \in [0, \infty)$ for every $x \in [1, \infty)$.

   (b) Prove that $\text{Rng}(f) = [0, \infty)$. [HINT: In light of part (a), you need only prove the other inclusion, $[0, \infty) \subseteq \text{Rng}(f)$. That is, for each $y \in [0, \infty)$, you must find some $x \in \text{Dom}(f) = [1, \infty)$ such that $f(x) = y$.]

   (c) Prove that f is an injection.

   (d) Conclude that f is a bijection from $[1, \infty)$ to $[0, \infty)$, and give a formula for the inverse function $f^{-1}\colon [0, \infty) \to [1, \infty)$.

   (e) Sketch the graph of f.

3. [**Falkner Section 11 Exercise 26**] Let A, B, and C be sets. Prove that if $f\colon A \to B$ and $g\colon B \to C$ are bijections, then $g \circ f\colon A \to C$ is a bijection.

4. [**Falkner Section 15 Exercise 1 – modified**] Show that the intervals $A = [1, \infty)$ and $B = (1, \infty)$ have the same cardinality by giving an example of a bijection $f\colon A \to B$.

   [HINT: Use one simple formula to define f on $\mathbb{N}$ and a different, even simpler formula to define f on $A \setminus \mathbb{N}$.]

   Be sure to prove that f is a bijection.

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. **[Falkner Section 11 Exercise 20 – modified]** Let

$$g\colon [0,1) \to \mathbb{R} \qquad\qquad h\colon (-1,0) \to \mathbb{R}$$
$$x \mapsto \frac{x}{1-x}. \qquad\qquad x \mapsto \frac{x}{1+x}.$$

   (a) Prove that $\text{Rng}(g) = [0,\infty)$ and $\text{Rng}(h) = (-\infty, 0)$.

   (b) Prove that both g and h are injections.

   (c) Conclude that g is a bijection from $[0,1)$ to $[0,\infty)$ and that h is a bijection from $(-1,0)$ to $(-\infty,0)$.

   (d) Find formulas for $g^{-1}\colon [0,\infty) \to [0,1)$ and $h^{-1}\colon (-\infty,0) \to (-1,0)$.

2. **[Falkner Section 11 Exercise 23]** Let

$$\varphi\colon (-1,1) \to \mathbb{R}$$
$$x \mapsto \frac{x}{1-|x|}.$$

   (a) Show that φ is a bijection from $(-1,1)$ to $\mathbb{R}$.

   (b) Find a formula for $\varphi^{-1}\colon \mathbb{R} \to (-1,1)$.

   [HINT: Use Practice Problem 1 above.]

3. **[Falkner Section 15 Exercises 6 & 7 – modified]**

   (a) Show that the intervals $[0,1)$ and $(0,1]$ have the same cardinality by giving an example of a bijection $f\colon [0,1) \to (0,1]$.

   (b) Show that the intervals $(0,1]$ and $(0,1)$ have the same cardinality by giving an example of a bijection $g\colon (0,1] \to (0,1)$.

   (c) Show that the intervals $[0,1]$ and $[0,1)$ have the same cardinality by giving an example of a bijection $h\colon [0,1] \to [0,1)$.

   (d) Conclude that the four intervals $[0,1]$, $[0,1)$, $(0,1]$, and $(0,1)$ all have the same cardinality.

   (e) Use the functions f, g, and h to construct a bijection from $[0,1]$ to $(0,1)$. [HINT: Use Exercise 3 above.]