# Homework 7

## Math 3345 – Autumn 2022 – Kutler

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. **[Falkner Section 3 Exercise 7]** Let $P$ be the sentence

$$(\exists x \in \mathbb{R})\big(x \geq 0 \text{ and } \sqrt{x+2} < \sqrt{x} + \sqrt{2}\big).$$

   (a) Use one of the generalized De Morgan's laws and one of the ordinary De Morgan's laws to show that $\neg P$ is logically equivalent to

$$(\forall x \in \mathbb{R})\big(x < 0 \text{ or } \sqrt{x+2} \geq \sqrt{x} + \sqrt{2}\big).$$

   (b) Is $P$ true or false? Provide a proof for your answer.

2. **[Falkner Section 3 Exercise 14]** For each of the following sentences, write out what it means in words, state whether it is true or false, and prove your answer.

   (a) $(\exists!\, x \in \mathbb{R})(2x + 7 = 3)$.

   (b) $(\exists!\, x \in \mathbb{R})(x^2 - 4x + 3 < 0)$.

   (c) $(\exists!\, x \in \mathbb{Z})(x^2 - 4x + 3 < 0)$.

   (d) $(\exists!\, x \in \mathbb{R})(x^2 - 4x + 4 = 0)$.

   (e) $(\exists!\, x \in \mathbb{R})(x^2 - 4x + 5 = 0)$.

   (f) $(\forall x \in \mathbb{R})(\exists!\, y \in \mathbb{R})(x + y = 0)$.

   (g) $(\forall x \in \mathbb{R})(\exists!\, y \in \mathbb{R})(xy = 1)$.

   (h) $(\forall x \in \mathbb{R})[\text{if } x \neq 0, \text{ then } (\exists!\, y \in \mathbb{R})(xy = 1)]$.

   (i) $(\forall x \in \mathbb{R})(\exists!\, y \in \mathbb{R})(xy = 0)$.

   (j) $(\forall x \in \mathbb{R})[\text{if } x \neq 0, \text{ then } (\exists!\, y \in \mathbb{R})(xy = 0)]$.

3. **[Falkner Section 5 Exercise 2]** Prove by induction that for each $n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

## Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. [**Falkner Section 3 Exercise 8**] Which of the variables $x$ and $y$ is free in the sentence $P(x, y)$. Answer the same question about each of the four sentences $(\exists y)P(x, y)$, $(\forall x)(\exists y)P(x, y)$, $(\forall x)P(x, y)$, and $(\exists y)(\forall x)P(x, y)$.

2. Read the (incorrect) proof given in [**Falkner Section 5 Exercise 8**], which purports to prove by induction that all horses have the same color.

   (a) Modify this proof to obtain (incorrect) proofs of the following (false) statements:

      i. All students at OSU have the same favorite food.

      ii. All songs are in the same key.

      iii. All rivers flow in the same direction.

      iv. All mountains have the same height.

   (b) Each of these proofs makes the same error. What is it?