Exam 1 Practice Problems

1. Show that $(P \wedge Q) \vee R$ is logically equivalent to $(P \vee R) \wedge (Q \vee R)$ in two ways
   (a) By using a truth table;
   (b) By giving an explanation in words.

2. Below, you are asked to determine if two sentences are logically equivalent. If the answer is **yes**, provide a proof (either in words or by using a truth table). If the answer is **no**, demonstrate this by choosing appropriate truth values for P, Q, R, and provide a brief justification.
   (a) Is $P \Rightarrow (Q \Rightarrow R)$ logically equivalent to $(P \Rightarrow Q) \Rightarrow R$?
   (b) Is $P \Rightarrow (Q \Rightarrow R)$ logically equivalent to $Q \Rightarrow (P \Rightarrow R)$?
   (c) Is $(P \wedge Q) \Rightarrow R$ logically equivalent to $(P \Rightarrow R) \vee (Q \Rightarrow R)$?
   (d) Is $(P \vee Q) \Rightarrow R$ logically equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$?
   (e) Is $\neg\big[(P \Rightarrow Q) \wedge P\big]$ logically equivalent to $(Q \Rightarrow P) \vee \neg P$?

3. For each sentence below, determine if the sentence is **always true** (i.e., it is a tautology) or if it is **possibly false**. If it is always true, provide a proof (either in words or by using a truth table). If it is possibly false, give truth values for P, Q, R making the sentence false, and provide a brief justification.
   (a) $P \wedge \neg P$
   (b) $P \vee \neg P$
   (c) $(P \wedge Q) \Rightarrow (P \vee Q)$
   (d) $(P \vee Q) \Rightarrow (P \wedge Q)$
   (e) $P \Rightarrow (Q \Rightarrow P)$
   (f) $(P \Rightarrow Q) \Rightarrow P$
   (g) $\big[(P \Rightarrow Q) \wedge \neg Q\big] \Rightarrow \neg P$
   (h) $\big[(P \Rightarrow Q) \wedge (Q \Rightarrow R)\big] \Rightarrow (P \Rightarrow R)$

4. In class, we saw that $\big[(P \Rightarrow Q) \wedge P\big] \Rightarrow Q$ is a tautology, called *modus ponens*.
   (a) Show that $(P \Rightarrow Q) \wedge P$ is logically equivalent to $P \wedge Q$.
   (b) Show that the converse of modus ponens is not a tautology. That is, find truth values for P and Q so that the sentence $Q \Rightarrow \big[(P \Rightarrow Q) \wedge P\big]$ is false.

5. Show that each of the following conditional sentences is a tautology by writing a conditional proof.

   (a) $P \Rightarrow (P \vee Q)$
   (b) $\big[P \Rightarrow (Q \wedge \neg Q)\big] \;\Rightarrow\; \neg P$
   (c) $\big[(P \Rightarrow \neg Q) \wedge (R \Rightarrow Q)\big] \;\Rightarrow\; (P \Rightarrow \neg R)$
   (d) $\big\{\big[(P \Rightarrow Q) \wedge (R \Rightarrow S)\big] \wedge (\neg Q \vee \neg S)\big\} \;\Rightarrow\; (\neg P \vee \neg R)$

6. Below, you are asked to determine if two sentences are logically equivalent. If the answer is **yes**, provide a proof (either in words or by using a truth table). If the answer is **no**, demonstrate this by giving a set A and component sentences $P(x)$ and $Q(x)$ making it false, and provide a brief justification.

   (a) Is
   $$(\exists x \in A)\big(P(x) \wedge Q(x)\big)$$
   logically equivalent to
   $$\big((\exists x \in A)P(x)\big) \wedge \big((\exists x \in A)Q(x)\big)?$$
   (b) Is
   $$(\forall x \in A)\big(P(x) \wedge Q(x)\big)$$
   logically equivalent to
   $$\big((\forall x \in A)P(x)\big) \wedge \big((\forall x \in A)Q(x)\big)?$$
   (c) Is
   $$(\exists x \in A)\big(P(x) \Rightarrow Q(x)\big)$$
   logically equivalent to
   $$\big((\exists x \in A)P(x)\big) \Rightarrow \big((\exists x \in A)Q(x)\big)?$$

7. For each of the following sentences, write out what it means in words, state whether it is true or false, and prove your statement.

   (a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 0)$
   (b) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 0)$
   (c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 20)$
   (d) $(\forall x \in \mathbb{R})\big[(x \neq 0) \Rightarrow (\exists y \in \mathbb{R})(xy = 20)\big]$
   (e) $(\forall x \in \mathbb{R})\big[(x \neq 0) \Rightarrow (\exists! y \in \mathbb{R})(xy = 20)\big]$
   (f) $(\forall m \in \mathbb{Z})\big[(m \neq 0) \Rightarrow (\exists! n \in \mathbb{Z})(mn = 20)\big]$
   (g) $(\forall m \in \mathbb{Z})(\exists n \in \mathbb{Z})(m < n)$
   (h) $(\exists n \in \mathbb{Z})(\forall m \in \mathbb{Z})(m < n)$

8. Prove the following statements using mathematical induction. Be sure to clearly state the inductive hypothesis, and explain what you are proving in the inductive step.

   (a) For every $n \in \mathbb{N}$,
   $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

   (b) For every $n \in \mathbb{N}$,
   $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (c) For every $n \in \mathbb{N}$,
   $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

   (d) For every $n \in \mathbb{N}$,
   $$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

   (e) For every $n \in \mathbb{N}$ such that $n > 3$, $n! > 2^n$.

   (f) For every $n \in \mathbb{N}$ such that $n > 6$, $n! > 3^n$.