Exam 2 Practice Problems

1. Prove the following.
   (a) The sum of two odd integers is even.
   (b) The sum of an even and an odd integer is odd.
   (c) The sum of two even integers is even.
   (d) The product of two odd integers is odd.
   (e) The product of an even integer and an odd integer is even.
   (f) The product of two even integers is even.

2. Let $n, m \in \mathbb{Z}$. Prove the following.
   (a) If nm is odd, then n is odd and m is odd.
   (b) If nm is even, then n is even or m is even.
   (c) If n^2 is odd, then n is odd.
   (d) If n^2 is even, then n is even.

3. Let $x, y \in \mathbb{R}$. Prove the following.
   (a) If x and y are rational, then $x + y$ is rational.
   (b) If x and y are rational, then xy is rational.
   (c) If y is rational and $y \neq 0$, then $1/y$ is rational.
   (d) If x and y are rational and $y \neq 0$, then x/y is rational.
   (e) If x is rational and y is irrational, then $x + y$ is irrational.
   (f) If x is rational and y is irrational, then xy is irrational.
   (g) If y is irrational, then $1/y$ is irrational. (Why is $y \neq 0$?)
   (h) If $x \neq 0$ is rational and y is irrational, then x/y is irrational.

4. Give examples to prove the following statements.
   (a) There exist irrational numbers x and y such that $x + y$ is irrational.
   (b) There exist irrational numbers x and y such that $x + y$ is rational.
   (c) There exist irrational numbers x and y such that xy is irrational.
   (d) There exist irrational numbers x and y such that xy is rational.

5. Prove the following.

   [HINT: Use the fact that any rational number can be written in lowest terms.]

   (a) $\sqrt{2}$ is irrational.
   (b) $\sqrt{3}$ is irrational.
   (c) $\sqrt{6}$ is irrational.
   (d) $\sqrt{2} + \sqrt{3}$ is irrational.

6. Let $d, n \in \mathbb{N}$. Use the definition of divisibility to show that if $d|n$, then $d \leq n$.

7. Let $a, b \in \mathbb{Z}$. Use the definition of divisibility to show that if $a|b$, then $a^2|b^2$.

8. Let a, b, q, r be integers such that $a = bq + r$. Prove that $\gcd(a, b) = \gcd(b, r)$.

9. Let $d \in \mathbb{N}$ and $n \in \mathbb{Z}$. Show that if $d|n$ and $d|(n + 1)$, then $d = 1$.

10. Let P be the sentence

    For all $a, b \in \mathbb{Z}$, if $a|b$ then $a|(b + 5a^2)$.

    Let Q be the sentence

    For all $a, b \in \mathbb{Z}$, if $a|b$ then $b + 5a^2$ is not prime.

    (a) Is the sentence P true? If so, provide a proof. If not, provide a counterexample.
    (b) Is the sentence Q true? If so, provide a proof. If not, provide a counterexample.

11. Use the Euclidean algorithm to compute $\gcd(84, 135)$.

12. (a) Use the Euclidean algorithm to compute $\gcd(30, 72)$.
    (b) Find integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 6$.
    (c) Do there exist integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 48$?
    (d) Do there exist integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 16$?

13. Find integers x and y such that $162x + 31y = 1$.

14. Use the prime factorizations

    $$3{,}219{,}398 = 2 \cdot 7^3 \cdot 13 \cdot 19^2 \qquad \text{and} \qquad 158{,}184 = 2^3 \cdot 3^2 \cdot 13^3$$

    to find gcd(3,219,398 , 158,184). Explain your reasoning.

15. (a) Let $x \in \mathbb{Z}$ and let p be a prime number. Prove that if p does not divide x, then $\gcd(p, x) = 1$.

    (b) Show that there exists $x \in \mathbb{Z}$ such that 12 does not divide x and $\gcd(12, x) \neq 1$. Why does this not contradict the result of part (a)?

16. Let n be an even integer. Prove that there exist unique integers $q, r \in \mathbb{Z}$ such that

$$n = 6q + r$$

and $r \in \{0, 2, 4\}$.

17. (a) Fill in the blanks: According to the division algorithm, when we divide an integer n by 5, we obtain unique integers $q, r \in \mathbb{Z}$ such that

$$n = \underline{\hspace{3cm}}$$

and

$$\underline{\hspace{1cm}} \leq r \leq \underline{\hspace{1cm}}.$$

    (b) Use the statement in part (a) to prove the following: For any integer $a \in \mathbb{Z}$, if $5|a^2$, then $5|a$.
    [HINT: Apply part (a) to $n = a^2$ and to $n = a$.]

18. Let $a \in \mathbb{N}$ and let p be a prime number. Prove that if $p|a^2$, then $p|a$.

    [HINT: Use unique prime factorization.]

19. Let $m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$. Prove that if

$$a \equiv b \mod m \qquad \text{and} \qquad c \equiv d \mod m,$$

then

$$a - c \equiv b - d \mod m.$$

20. Without using a calculator, find the natural number k such that $0 \leq k \leq 14$ and k satisfies the given congruence.

    (a) $2^{75} \equiv k \mod 15$
    (b) $6^{41} \equiv k \mod 15$
    (c) $140^{874} \equiv k \mod 15$

21. Without using a calculator, show that 15 divides $37^{42} - 38^{90}$.

22. (a) Check that $r^3 \equiv r \mod 6$ for every integer r such that $0 \leq r \leq 5$.
    (b) Use part (a) to prove that $n^3 \equiv n \mod 6$ for every integer n.
    (c) If x is a real number such that $x^3 = x$, then either $x = 0$ or we can divide by x to get $x^2 = 1$ (from which we conclude $x = 1$ or $x = -1$).
    Given the result of part (b), we might wonder if similar reasoning implies that for every integer n, either $n \equiv 0 \mod 6$ or $n^2 \equiv 1 \mod 6$. Is this true?

23. Prove that

$$7^n \equiv 1 + 6n \mod 9$$

for every $n \in \mathbb{N}$.

24. Make addition and multiplication tables for arithmetic
    (a) modulo 2.
    (b) modulo 3.
    (c) modulo 4.
    (d) modulo 5.