Induction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of <u>natural numbers</u>.

<u>Ex</u>: You might have seen the following formula in Calc II:

For each $n \in \mathbb{N}$,

$$\underbrace{1 + 2 + 3 + \cdots + n}_{= \sum_{i=1}^{n} i} = \frac{n(n+1)}{2}$$

How can we prove $(\forall n \in \mathbb{N})\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$?

We need an argument that works for <u>every</u> n — but as n gets larger we get more and more summands.

<u>$n=1$</u>: $1 = \frac{1 \cdot 2}{2}$

<u>$n=2$</u>: $1 + 2 = 3 = \frac{2 \cdot 3}{2}$

<u>$n=3$</u>: $1 + 2 + 3 = 6 = \frac{3 \cdot 4}{2}$

etc.

The Principle of Mathematical Induction

Let $P(n)$ be a sentence involving $n \in \mathbb{N}$.

Ex: $P(n) =$ " $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ "

The PoMI is: Suppose

① $P(1)$ is true [base case]

and

② For any $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n+1)$ is true. [inductive step]

Then $P(n)$ is true for every $n \in \mathbb{N}$.

In symbols:

$$\left\{P(1) \wedge \left[(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))\right]\right\} \Rightarrow (\forall n \in \mathbb{N})\, P(n)$$

The PoMI is an axiom of the integers. It is assumed to be true by definition.

Why should we accept it?

Intuition – dominoes, trains, etc.

Thm: For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: Let $P(n)$ be "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$".
We will prove $(\forall n \in \mathbb{N})\, P(n)$ by induction on n.

Base Case: When $n = 1$,

$$\sum_{i=1}^{1} i = 1 \quad \text{and} \quad \frac{1(1+1)}{2} = 1,$$

so $P(1) =$ "$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$" is true. ✓

Inductive Step: Let $n \in \mathbb{N}$. We will prove $P(n) \Rightarrow P(n+1)$ is true.

Suppose $P(n)$ is true. That is,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for this particular number n.

So $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Add $n+1$ to both sides:

$$
\begin{aligned}
1+2+3+\cdots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)[(n+1)+1]}{2}.
\end{aligned}
$$

$\sum_{i=1}^{n+1} i$

This is precisely $P(n+1)$. So $P(n+1)$ is true, completing the inductive step.

We conclude, by mathematical induction, that $P(n)$ is true for every $n \in \mathbb{N}$. ∎

Thm: For every $n \in \mathbb{N}$,

$$\underbrace{1+3+5+\cdots+(2n-1)}_{=\sum_{i=1}^{n}(2i-1)} = n^2.$$

Proof: We proceed by induction on n.

Let $P(n)$ be "$1+3+5+\cdots+(2n-1)=n^2$."

Base Case: When $n=1$, $P(1)$ is

"$1 = 1^2$"

which is true.

Inductive Step: Let $n \in \mathbb{N}$. We wish to prove $P(n) \Rightarrow P(n+1)$, so we may assume $P(n)$.

Thus,

$$1+3+5+\cdots+(2n-1)=n^2$$

is true (for this n).

Now,

$$1+3+5+\cdots+(2n-1)+[2(n+1)-1]$$
$$= n^2 + [2n+2-1]$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2.$$

Thus, we have shown that $P(n+1)$ is true, completing the inductive step.

By induction, we conclude that $P(n)$ is true for all $n \in \mathbb{N}$. $\blacksquare$

Does the base case have to be $n=1$?

No!

Ex (Exam I Review 8(e)): For every $n \in \mathbb{N}$ such that $n > 3$, $2^n < n!$

Check: $2^3 = 8 > 6 = 3!$ ✗
$2^4 = 16 < 24 = 4!$ ✓
$2^5 = 32 < 120 = 5!$ ✓

Proof: Let $P(n)$ be "$2^n < n!$"
We will show $P(n)$ is true for every $n \in \mathbb{N}$ with $n > 3$ by induction.

Base Case: $n = 4$. Since $2^4 = 16$ and $4! = 24$, $P(4)$ is true.

Inductive Step: Let $n \in \mathbb{N}$ such that $n > 3$. We must prove $P(n) \Rightarrow P(n+1)$.

Assume $P(n)$ is true, so $2^n < n!$
Multiply by 2 to get

$$\underbrace{2 \cdot 2^n}_{= 2^{n+1}} < 2 n!$$

Since $2 < 3 < n < n+1$, we have

$$2^{n+1} < 2\,n! < (n+1)\,n! = (n+1)!$$

Thus, $P(n+1)$ is true, completing the inductive step.

We conclude that $P(n)$ is true for every $n \in \mathbb{N}$ such that $n > 3$. $\blacksquare$

<u>Note</u>: We could have equivalently set

$$Q(n) = P(n+3) = \text{“} 2^{n+3} < (n+3)! \text{”}$$

and proved $(\forall n \in \mathbb{N})\, Q(n)$ by induction starting at $n = 1$.