Warm-Up: Prove or disprove:

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers in lowest terms, then

$$\frac{ad+bc}{bd}$$

is also in lowest terms.

Irrational Numbers

Def: Let $x \in \mathbb{R}$. We say x is irrational if $x \notin \mathbb{Q}$.

That is, for all $a, b \in \mathbb{Z}$ with $b \neq 0$, $x \neq \frac{a}{b}$.

Ex:
- $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$ are irrational
- $\sqrt{n}$ is irrational if $n \in \mathbb{Z}$ is not a perfect square.
- $\sqrt[3]{2}$ is irrational
- π and e are irrational

Hard to prove any of these!
- π Lambert, 1761
- e Euler, 1731

To show x is irrational, we assume it is rational and get a contradiction.

Thm: Let $x \in \mathbb{Q}$ and let $y \in \mathbb{R}$ be irrational.

① $x+y$ is irrational.

② If $x \neq 0$, then $x \cdot y$ is irrational

Proof: ① Suppose, to get a contradiction, that $x+y \in \mathbb{Q}$. Since x is rational, $-x$ is rational (HW 13). Thus,

$$y = (x+y) + (-x)$$

is the sum of two rational numbers, so $y \in \mathbb{Q}$, a contradiction.

② HW 14.

What about the sum of two irrational numbers?

- It can be rational: $\sqrt{2}$ is irrational. So is $-\sqrt{2} = (-1) \cdot \sqrt{2}$. But $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$.

- It can be irrational: $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ is irrational

$$\underset{\text{rational}}{2}\ \underset{\text{irrational}}{\sqrt{2}}$$

The same thing happens with multiplication:

$$\underset{\text{irr.}}{\sqrt{2}} \cdot \underset{\text{irr.}}{\sqrt{2}} = 2 \in \mathbb{Q} \qquad \underset{\text{irr.}}{\sqrt{2}} \cdot \underset{\text{irr.}}{\sqrt{3}} = \sqrt{6} \notin \mathbb{Q}$$

Let's prove that $\sqrt{2}$ is irrational. We'll use

Fact: Let $n \in \mathbb{Z}$. If n^2 is even, then n is even.
(HW 9)

Thm: For every $x \in \mathbb{Q}$, $x^2 \neq 2$.

This actually only shows $\sqrt{2} \notin \mathbb{Q}$. To prove that $\sqrt{2}$ is a real number, you need to use the Least Upper Bound Property.

Proof: Suppose, to get a contradiction, that there is some $x \in \mathbb{Q}$ such that $x^2 = 2$.

Let $x = \frac{a}{b}$ be the representation of x in lowest terms, where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

This means: If $d \in \mathbb{N}$ and $d \mid a$ and $d \mid b$, then $d = 1$.

We have $x^2 = \left(\frac{a}{b}\right)^2 = 2$, so $\frac{a^2}{b^2} = 2$.
Therefore,

$$a^2 = 2b^2. \qquad (*)$$

Since $b^2 \in \mathbb{Z}$, this shows a^2 is even, and thus a is even as well.

Then $a = 2k$ for some $k \in \mathbb{Z}$. Now $(*)$ becomes

$$(2k)^2 = 2b^2$$
$$4k^2 = 2b^2.$$

We may divide both sides by 2 (or use Multiplicative Cancellation) to get

$$2k^2 = b^2.$$

But this means b^2 is even, and thus so is b.

Now $2|a$ and $2|b$, which contradicts $x = \frac{a}{b}$ being in lowest terms.

We conclude that there is no such x in $\mathbb{Q}$. $\blacksquare$

A stronger statement

We'll soon prove the following important fact:

> **Thm:** Let p be a prime and $x, y \in \mathbb{Z}$. Then
> $$p \mid xy \implies p \mid x \text{ or } p \mid y$$

If we assume this for the moment, we can prove the following:

Cor (Book Example 4.52): If $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$, then $x \in \mathbb{Z}$.

↳ Take $x = \sqrt{n}$ for $n \in \mathbb{N}$. Then this proves that either

- $\sqrt{n} \in \mathbb{N}$, i.e. n is a perfect square

or

- $\sqrt{n}$ is irrational.

Proof of Cor: Suppose $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$.

Write $x = \frac{a}{b}$ in lowest terms with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

Our goal is to show $b=1$, so $x=a\in\mathbb{Z}$.

Let's assume $b\neq 1$ and get a contradiction.

Since $b\neq 1$ and $b\in\mathbb{N}$, we have $b>1$. Thus, there is some prime p such that $p\mid b$.

Now, $x^2=\frac{a^2}{b^2}=n$ for some $n\in\mathbb{Z}$. Thus,

$$a^2=b^2n=b(bn).$$

That is, $b\mid a$. By transitivity of divisibility, $p\mid a$ also. But this contradicts the fact that $\frac{a}{b}$ is in lowest terms.

Therefore, we conclude $b=1$ and thus $x\in\mathbb{Z}$. ∎